Model Category Structures on $s\mathcal{C}$

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Abstract

We study the different model category structures on the simplicial objects over a model category $\mathcal{C}$, with a goal of constructing a simplicial model category structure on this category. We first consider the Reedy model category structure, which fails to make the category into a simplicial model category. By restricting our view to complete and cocomplete pointed model categories in which every object is fibrant, we are able to define a model category structure, called the E^2 model category structure that makes simplicial objects over $\mathcal{C}$ into a simplicial model category. For both the Reedy and the E^2 constructions we compute the structure in the example of chain complexes over a ring R.

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1 Introduction

in 1967 Daniel Quillen introduced a category-theoretic construction which made it possible to define homotopy on a category; he called the construction a model category structure. A model category structure is defined to be three classes of morphisms — called fibrations, cofibrations and weak equivalences — that satisfy several properties analogous to conditions ontheme maps in topological spaces. Any model category structure on a category $\mathcal{C}$ yeilds a homotopy theory; for example, on topological spaces there are two such structures, one producing the standard homotopy theory, and one producing rational homotopy theory. These structures are also helpful in proving that certain model structures are equivalent; for example, one of the connections that model category structures make clear is the equivalence between the homotopy theory of topological spaces and that of simplicial sets.

In addition to simplicial sets, we can look at other kinds of simplicial objects. We can consider a simplicial object over a category $\mathcal{C}$ to have an object in every dimension (corresponding to the "n-simplices" of the simplicial object) and maps between them that satisfy the same relations as the face and degeneracy maps of a simplicial set.

A simplicial category is a category along with a notion of a "mapping space." Instead of the morphisms between objects being represented as just the set of maps between them, the mapping space is a simplicial set that encapsulates the idea of homotopies between the maps, so that, in particular, the connected components of the mapping space between two objects X and Y represents the homotopy classes of maps $X \to Y$. It turns out that if $\mathcal{C}$ is closed under arbitrary limits and colimits then $s\mathcal{C}$ is automatically a simplicial category, although not necessarily a model category.

If a category $\mathcal{C}$ is both a simplicial category and a model category, we can ask if these structures are compatible. For example, we want the map induced on mapping spaces by a fibration $X \to Y$ to also be a fibration. This kind of compatibility condition is encoded in an extra axiom, called "Axiom SM7." A category that satisfies SM7 is called a simplicial model category. Note that a category that is both a simplicial category and a model category is not necessarily a simplicial model category.

Now suppose that $\mathcal{C}$ is a model category. Can we then define a model category structure on $s\mathcal{C}$ that is related to the model category structure on $\mathcal{C}$? In an influential but unpublished paper, Reedy constructed such a model category structure, where in particular the weak equivalences were simply the dimensionwise weak equivalences. If $\mathcal{C}$ itself is a simplicial category then $s\mathcal{C}$ inherits a simplicial structure that is compatible with the Reedy model category structure, and therefore makes $s\mathcal{C}$ into a simplicial model category. However, the Reedy model category structure is *not* compatible with the standard simplicial structure defined on $s\mathcal{C}$. Thus with this structure $s\mathcal{C}$ becomes both a simplicial and a model category, but not a simplicial model category.

In order to remedy this, in 1992 Dwyer, Kan, and Stover introduced a different model category structure on $s\mathcal{C}$, called the E^2 model category structure. This structure is related to the Reedy structure, but has a larger set of weak equivalences (and correspondingly smaller classes of fibrations and cofibrations) that makes $s\mathcal{C}$ into a simplicial model category.

This paper has four parts. In the first part we give a short introduction to model categories and the homotopy theory associated to them. In the second, we do a quick overview of simplicial sets and simplicial categories. Then, in the third part, we proceed to construct the Reedy model category structure on $s\mathcal{C}$, and compute the structure on $s\mathbf{Ch}_R$, simplicial chain complexes over a ring R. Lastly, we will construct the E^2 model category structure, and also compute it for the example of $s\mathbf{Ch}_R$.

2 Model Categories

2.1 Definition of a Model Category

When studying topological spaces (and distinguishing them) there are commonly two invariants that are considered: homotopy and (co)homology. Both of these work in similar fashions: they take a geometric object, define an equivalence of two of these objects and an operation on them, and consider the group that is formed modulo the given equivalence. In general, there are many objects other than topological spaces that such operations are applied to: chain complexes, groups, etc. We therefore attempt to generalize this construction to a general categorical construction.

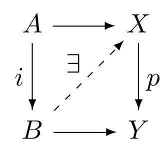
This is done through the introduction of a *model category structure*.

Definition 2.1. A *model category structure* is a category $\mathcal{C}$, closed under finite limits and colimits, together with three classes of maps, which we will call cofibrations, fibrations, and weak equivalences. These classes must satisfy the following four axioms:

MC1: If $f : A \to B$ and $g : B \to C$ are maps in $\mathcal{C}$, and two of f, g, gf are weak equivalences, then so is the third.

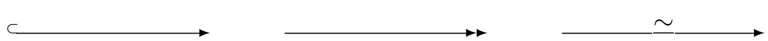
MC2: If f is a retract of g and g is a fibration, cofibration, or weak equivalence, then so is f.

MC3: In the following diagram, if i is a cofibration, p is a fibration, and at least one of i and p is a weak equivalence, then there exists a map $h : B \to X$ that makes the diagram commute.



MC4: Any map $f : A \to B$ can be factored into a cofibration followed by a fibration, where either the cofibration or the fibration is also a weak equivalence.

We will call a fibration that is also a weak equivalence an *acyclic fibration* and a cofibration that is also a weak equivalence an *acyclic coFibration*. We adopt the following notation:



cofibration fibration weak equivalence

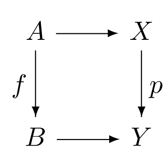
Notice that since $\mathcal{C}$ contains all finite limits and colimits it must contain both an initial object $\varnothing$ and a terminal object $*$.

Definition 2.2. We call an object X of $\mathcal{C}$ *fibrant* if $X \to *$ is a fibration. We call it *cofibrant* if $\varnothing \to X$ is a cofibration.

Examining the axioms governing fibrations, cofibrations, and weak equivalences we notice that if $\mathcal{C}$ has a model category structure then $\mathcal{C}^{\mathrm{op}}$ has the opposite model category structure with the fibrations and cofibrations switched. Thus in particular if we can prove something about cofibrations based only on the axioms then we get the dual statement about fibrations "for free."

It turns out that axioms defining the maps are actually somewhat redundant: only two of the three sets of weak equivalences, fibrations, and cofibrations are required to construct the third.

Definition 2.3. Let $\mathscr{A}$ be a class of morphisms of $\mathcal{C}$. A map $f : A \to B$ has the *left lifting property* (LLP) with respect to $\mathscr{A}$ if for any commutative diagram



with $p \in \mathscr{A}$, there exists a map $h : B \to X$, that makes the diagram commute.

Similarly, a map $g : A \to B$ has the *right lifting property* (RLP) with respect to $\mathscr{A}$ if for any commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & A \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \longrightarrow & B \end{array}$$

with $p \in \mathscr{A}$ there exists a map $h : Y \to A$ that makes the diagram commute.

Proposition 2.4.

1. *The set of cofibrations is exactly the set of maps that has the LLP with respect to the acyclic fibrations.*

2. *The set of acyclic cofibrations is exactly the set of maps that has the LLP with respect to the fibrations.*

3. *The set of fibrations is exactly the set of maps that has the RLP with respect to the acyclic cofibrations.*

4. *The set of acyclic fibrations is exactly the set of maps that has the RLP with respect to the cofibrations.*

Proof. Note that if we can show the first two statements then we are done, as we obtain the last two by dualization. In addition, notice that the forward directions are obvious by MC4. Thus it simply remains to show that a map can be characterized by the maps that it has lifting properties with respect to.

Suppose that $f : X \to Y$ has the LLP with respect to the acyclic fibrations. Factoring f into a cofibration followed by an acyclic fibration, we get the following commutative diagram:

$$\begin{array}{ccc} X & \overset{i}{\hookrightarrow} & \widetilde{Y} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \wr} \\ Y & \underset{}{\overset{\text{id}}{=\!=}} & Y \end{array}$$

Since f has the LLP with respect to all acyclic fibrations, a map $g : Y \to \widetilde{Y}$ that makes the diagram commute exists. In particular, this means that f is a retract of i, meaning that f is a cofibration.

Now suppose that $f : X \to Y$ has the LLP with respect to the fibrations. Factoring f into an acyclic cofibration followed by a fibration gives us (by an analogous argument to the one above) that f must be an acyclic cofibration. So we are done. □

In particular, this implies that cofibrations are preserved under pushouts and fibrations are preserved under pullbacks.

2.2 Example: Topological Spaces

We will show that we can define a model category structure on **Top**, the category of topological spaces and continuous maps. In particular, we will call a map $f : X \to Y \in$ **Top**

- a weak equivalence if it is a weak homotopy equivalence, meaning that f induces an isomorphism of groups $\pi_n(X, x) \to \pi_n(Y, f(x))$ for all $n \geq 1$, and is a bijection when $n = 0$,

- a fibration if it is a Serre fibration, meaning that it has the RLP with respect to the inclusions $D^n \to D^{n+1}$, and

- a cofibration if it has the LLP with respect to the acyclic fibrations.

Theorem 2.5. *With the above structure* **Top** *becomes a model category.*

Proof. We follow the proof in [3].

Axiom MC1 is clear from the definition of weak equivalence. MC2 is clear for weak equivalences, as π_n is a functor. Now suppose that $f : A \to B$ is a retract of a cofibration $f' : A' \to B'$. Given a diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle f}\big\downarrow & & \wr\big\downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

we can extend this to the left by the retract diagram for f, obtaining

$$\begin{array}{ccccccc} A & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & X \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle f'}\big\downarrow & & \exists \nearrow & & \wr\big\downarrow{\scriptstyle p} \\ B & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & Y \end{array}$$

where a lift exists because f' is a cofibration; this clearly allows us to construct a lift $B \to X$ that will make the diagram commute. The fibration case follows analogously.

We will first prove MC4; then we will apply it to prove MC3. In order to prove MC4 we need to show that any map $X \to Y \in$ **Top** can be factored into $X \xrightarrow{i} X' \xrightarrow{p} Y$ where i is a weak homotopy equivalence with the LLP with respect to the fibrations, and p is a fibration.

To prove this, let $\mathcal{F}$ be the set of maps $\{D^n \to D^{n+1}\}$. Let $Z \to Y \in$ **Top**; then $S_n Z$ is the set of diagrams $\begin{array}{ccc} D^n & \longrightarrow & Z \\ \big\downarrow & & \big\downarrow \\ D^{n+1} & \longrightarrow & Y \end{array}$. We define $\mathcal{S}(Z)$ to be the pushout of the diagram

$$Z \longleftarrow \coprod_{n\geq 0} \coprod_{S_n Z} D^n \longrightarrow \coprod_{n\geq 0} \coprod_{S_n Z} D^{n+1}$$

Note that there is a canonical map $\mathcal{S}Z \to Y$. This construction adds a cylinder to Z along one end for each commutative square with the left-hand side in $\mathcal{F}$; thus it is a deformation retract, and therefore a weak homotopy equivalence. The map $Z \to \mathcal{S}Z$ is also clearly a cellular map. We define

$$X' = \varinjlim_n \mathcal{S}^n X.$$

Then the map i will be a weak homotopy equivalence that has the LLP with respect to the fibrations (by definition). It remains to show that p is a fibration. Consider a commutative square

$$\begin{array}{ccc} D^n & \xrightarrow{g} & X' \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ D^{n+1} & \longrightarrow & Y \end{array}$$

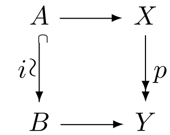
Since D is compact there must be an integer k such that g factors through $\mathcal{S}^k X$, so that we get a diagram

$$\begin{array}{ccccccc} D^n & \xrightarrow{g'} & \mathcal{S}^n X & \longrightarrow & \mathcal{S}^{n+1} X & \longrightarrow & X' \\ \big\downarrow & & \big\downarrow & \exists \nearrow & \big\downarrow & & \big\downarrow{\scriptstyle p} \\ D^{n+1} & \longrightarrow & Y & = & Y & = & Y \end{array}$$

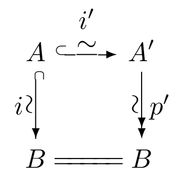
Notice that the left-hand diagram would have been in the construction for $S^{n+1}X$, so by construction the lift shown by the dashed line exists; thus p has the RLP with respect to all of the maps in $\mathcal{F}$, showing that it is a fibration.

The cofibration-acyclic fibration case is done analogously, only with $\mathcal{F}$ being the set of maps $S^{n-1} \to D^n$.

Lastly we prove MC3. Notice that if p is an acyclic fibration the statement is obvious by the definition of cofibration. We now simply need to show that if the cofibration is acyclic then there is a lift. Suppose we have a diagram



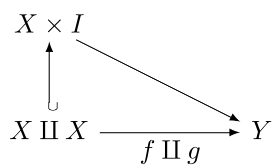
Factor i as an acyclic cofibration i' that has the LLP with respect to all fibrations followed by a fibration p'. By MC1 the fibration will also be a weak equivalence. Thus we have a diagram



By definition, a lift exists in this diagram. So, in particular, i is a retract of i'. Since i' has the left lifting property with respect to all fibrations, i must also. So we are done. ☐

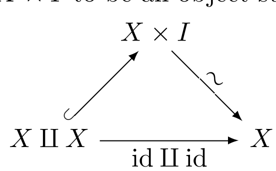
2.3 Properties of Model Categories

An important notion in distinguishing different maps between topological spaces is that of homotopy equivalence. A homotopy equivalence between maps $f, g : X \to Y$ is usually defined as a map from $X \times I$ to Y, where $X \times \{0\} \to Y$ is f and $X \times \{1\} \to Y$ is g. In particular, such a map is an interpolation between two given maps. So if we could define what such an "interpolation object" was, and what "restrictions" were, we could define a homotopy equivalence to be a map from such an interpolation object into the given space. We can rephrase the restriction condition by asserting that the following diagram commutes:



In particular, we state what the map must be on the two boundary copies of S^n. For a homotopy equivalence to exist all that is required is for a factoring through an "interpolation object" to exist.

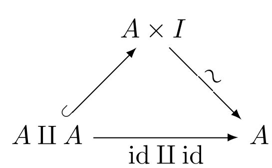
Notice that if we can define $X \times I$ in a category-theoretic fashion we would then have a passable definition of homotopy equivalence on objects defined on a category. In **Top**, what is $X \times I$? In the above context it is a cylinder which is homotopy equivalent to X which can have two disjoint copies of X embedded in it. In particular, it does not matter that we precisely crossed X with I; this is just a particularly nice case in **Top**. Thus in a general category we can define $X \times I$ to be an object such that the following diagram commutes:



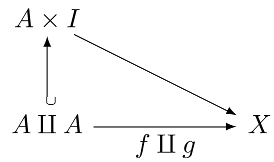
Note that (at least in the case of topological spaces) we encode both the fact that we have an inclusion of $X \amalg X$ and that $X \times I$ is weakly equivalent to X.

Thus we have come up with the following definition:

Definition 2.6. We define a *cylinder object* for A to be an object $A \times I$ such that the following diagram commutes:



We call two maps $f, g : A \to X$ *left-homotopic* if there exists a cylinder object $A \times I$ that makes the following diagram commute:

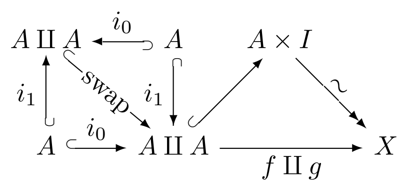


If f and g are left homotopic we write $f \overset{\ell}{\sim} g$.

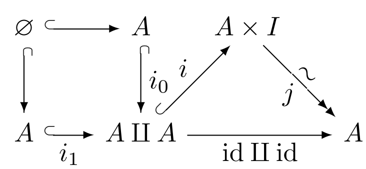
In general, $f \overset{\ell}{\sim} g$ is not an equivalence relation. However, if we only look at maps out of cofibrant objects, it becomes an equivalence relation.

Proposition 2.7. *If A is cofibrant then $\overset{\ell}{\sim}$ is an equivalence relation on* $\mathrm{Hom}(A, X)$ *for any objects X.*

Proof. We follow the proof in [3]. Note that if A is cofibrant then it can be defined to be its own cylinder object. Thus $f \overset{\ell}{\sim} f$. The following diagram shows symmetry:

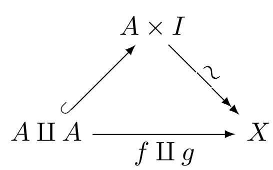


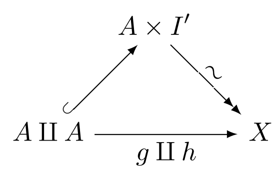
Thus it remains to show transitivity. Consider the following commutative diagram:



Notice that $i_0 i$ and $i_1 i$ are both acyclic cofibrations. (They are weak equivalences by MC1 because $j i_0 i = \mathrm{id}$, which is a weak equivalence.)

Suppose that we have $f \overset{\ell}{\sim} g$, with homotopy $H : A \times I \to X$ and $g \overset{\ell}{\sim} h$ with homotopy $K : A \times I' \to X$. so that we have the following two diagrams:

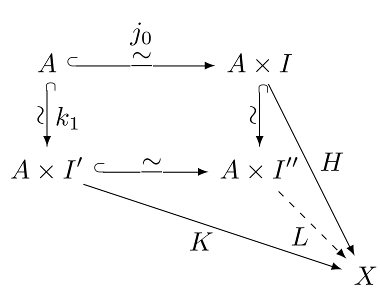




Let j_0, j_1 be the two maps $A \to A \times I$ and k_0, k_1 be the two maps $A \to A \times I'$. In particular, we know that

$$p_1 j_0 = f \qquad p_1 j_1 = g \qquad p_2 k_0 = g \qquad p_2 k_1 = h.$$

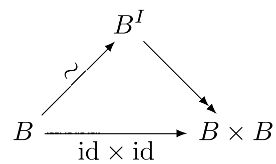
Notice that if we define $A \times I''$ to be the pushout of the diagram $A \times I \xleftarrow{j_0} A \xrightarrow{k_1} A \times I'$ we get the following picture



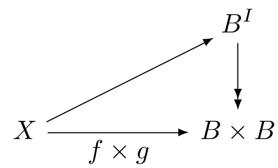
where L is the desired homotopy. Notice that $A \times I''$ is a cylinder object for A, since $A \times I'' \to A$ is a weak equivalence by MC1. □

Thus we have a notion of "homotopy equivalent" maps for cofibrant objects. However, our definition of homotopy equivalence is *not* the same under dualization as cofibrant objects map to fibrant ones. Since the model category structure is closed under dualization it should really be the case that the definition of homotopy equivalence should also be closed under dualization. This induces us to look at the dual definition of homotopy:

Definition 2.8. We define a *path object* B^I to be the object that makes the following diagram commute:



Two maps $f, g : X \to B$ are *right homotopic* if there exists a path object B^I such that the following diagram commutes:



Thus we also get the dual proposition:

Proposition 2.9. *If B is fibrant then $\overset{r}{\sim}$ is an equivalence relation on* $\mathrm{Hom}(X, B)$ *for all objects X.*

Proof. Simply take the dual of the proof for the first version of the proposition. □

We now have two different statements about homotopy equivalence. In particular, we would like to compare these notions on the hom-sets on which they are both equivalence relations; if the definitions work out the way we want them to, they would be equivalent on the sets where they are both equivalence relations. (Indeed, on topological spaces they are equivalent: if we have a homotopy $f : X \times I \to Y$ it is "equivalent" to a homotopy $g : X \to Y^I$ by letting $g(x)(t) = f(x,t)$.)

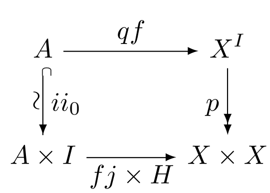
It turns out that this is indeed the case:

Proposition 2.10. *If A if cofibrant and B is fibrant, and $f, g : A \to B$ then*

$$f \overset{\ell}{\sim} g \Longleftrightarrow f \overset{r}{\sim} g.$$

Proof. We follow the proof in [3]. Note that it suffices to show that $f \overset{\ell}{\sim} g$ implies $f \overset{r}{\sim} g$ (we get the other direction by dualization).

Since $f \overset{\ell}{\sim} g$ we have a homotopy $H : A \times I \to X$ such that $Hi = f \amalg g$. Let $X \overset{q}{\xrightarrow{\sim}} X^I \overset{p}{\twoheadrightarrow} X \times X$ be a construction of a path object. We want to find a homotopy $K : A \to X^I$ such that $pK = f \times g$. Notice that we can apply MC3 to



to get a map $K : A \times I \to X^I$. We claim that Kii_1 is exactly the map we want. From the definition of H we know that $Hii_0 = f$ and $Hii_1 = g$. Thus

$$pKii_1 = (fj \times H)ii_1 = f \times Hii_1 = f \times g$$

as desired. Thus $f \overset{\ell}{\sim} g \Rightarrow f \overset{r}{\sim} g$. By duality, we get the converse. □

Notice that the proof of the proposition implies the following corollary:

Corollary 2.11. *Let $f, g : X \to Y$. If $f \overset{\ell}{\sim} g$ then for any choice of path object Y^I, $f \overset{r}{\sim} g$ with respect to that path object. Dually, if $f \overset{r}{\sim} g$ then for any choice of cylinder object $X \times I$, $f \overset{\ell}{\sim} g$ with respect to that cylinder object.*

So, for maps between cofibrant and fibrant objects we can define a simple notion of "homotopy equivalence": $f \sim g$ when $f \overset{\ell}{\sim} g$ and $f \overset{r}{\sim} g$. In addition, we only need one "cylinder object" and one "path object" for every object, as the set of homotopies with respect to those objects is the same as the set of homotopies for all path/cylinder objects. This definition is also unchanged under dualization, which is consistent with the fact that the opposite of a model category is also a model category.

Now consider $\mathcal{C}_{CF}$, the full subcategory of $\mathcal{C}$ consisting of only the cofibrant and fibrant objects. On this subcategory, homotopy equivalence between maps, and homotopy equivalence of compositions are well-defined.

Definition 2.12. The *homotopy category* $\mathrm{Ho}\,\mathcal{C}$ is the category whose objects are the cofibrant fibrant objects of $\mathcal{C}$ and where the morphisms are homotopy classes of maps in $\mathcal{C}$.

Now we come to the major theorem of model categories: that the homotopy category exists, and that any functor mapping weak equivalences to isomorphisms can be factored through it.

Theorem 2.13. *The category $\mathrm{Ho}\,\mathcal{C}$ exists for all model categories $\mathcal{C}$. There exists a functor $\gamma : \mathcal{C} \to \mathrm{Ho}\,\mathcal{C}$ which sends an object to a cofibrant fibrant approximation, and any weak equivalence to an isomorphism. In addition, any functor $\mathcal{C} \to \mathcal{D}$ that sends weak equivalences to isomorphisms factors through γ.*

For a proof of this theorem see [3].

2.4 Another Example: $\mathbf{Ch}_R$

Definition 2.14. We define $\mathbf{Ch}_R$ to be the category of nonnegatively graded chain complexes over a ring R.

We call a map $f : M \to N \in \mathbf{Ch}_R$

- a weak equivalence if it induces an isomorphism $H_k(M) \to H_k(N)$ for all $k \geq 0$,
- a fibration if it is an epimorphism in dimensions strictly greater than zero, and
- a cofibration if the map $f_k : M_k \to N_k$ is injective with projective cokernel for all $k \geq 0$.

Theorem 2.15. *The structure above is a model category structure on $\mathbf{Ch}_R$.*

Proof. We follow the proof in [3].

Axiom MC1 is obvious. Axiom MC2 follows from the fact that H is a functor, and the fact that in the category of modules over R, a retract of a monomorphism or an epimorphism is also a monomorphism or epimorphism (respectively), and that a retract of a projective module is projective.

Now consider axiom MC3. Consider a diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & X \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle \wr\, p} \\ B & \longrightarrow & Y \end{array}$$

We know that p_k is an epimorphism for $k > 0$. Notice that p_0 must also be an epimorphism, as p induces an isomorphism in 0-th homology. Since p is onto we can put it into a short exact sequence

$$0 \longrightarrow K \longrightarrow X \xrightarrow{p} Y \longrightarrow 0,$$

with the homology of K trivial in each dimension.

We construct the required map $h : B \to X$ by induction. We know that $B_0 \cong A_0 \oplus P_0$, where P_0 is projective. Thus we can define h_0 to be equal to g_0 on A_0 and to be a lift of $P_0 \to Y$ to X on P_0 (which exists because P_0 is projective). Now suppose that we have a lift h up to dimension $n-1$. Then we can write $B_n \cong A_n \oplus P_n$, and we can use the same construction as for the zero case to make a function $h_n : B_n \to X_n$. Notice that h_n makes the diagram commute in level n, but does not necessarily commute properly with the differential ∂_{n-1}. Let $\epsilon : B_n \to X_{n-1}$ be the map $\partial_{n-1} h_n - h_n \partial_{n-1}$. We want ϵ to be the zero map. Note that $\partial_{n-2}\epsilon = 0$, $p_{n-1}\epsilon = 0$ and $\epsilon i_n = 0$, since the map was a chain map up to h_{n-1} and as the horizontal maps commute with the differential. But this means that ϵ induces a map

$$\mathcal{E} : B_n / i_n(A_n) \cong P_n \to K_{n-1}$$

whose image will consist only of cycles. Since K is acyclic ∂_{n-1} will be an epimorphism onto the image of $\mathcal{E}$, which means (as P_n is projective) that there will be a lift $\mathcal{E}' : P_n \to K_n$, which then gives us a map $\mathcal{E}'' : B_n \to X_n$. The map $h_n - \mathcal{E}''$ is the desired map, completing the induction.

Before we prove the second half of MC3 we will prove a lemma.

Lemma 2.16. *We define $D_n(A)$ to be the complex that is 0 in every dimension except for n and $n-1$, and is equal to A in those dimensions (with ∂ being the identity).*

Let $P \in \mathbf{Ch}_R$ be acyclic and such that each P_n is projective. Then each module $\ker \partial_k$ is projective, and

$$P \cong \bigoplus_{k \geq 1} D_k(\ker \partial_{k-1}).$$

Proof. We define a complex P^k by

$$P^k_n = \begin{cases} P_n & \text{if } n > k-1 \\ \operatorname{im} \partial_{k-1} & \text{if } n = k-1 \\ 0 & \text{otherwise} \end{cases}$$

Since P is acyclic we know that $P^k / P^{k+1} \cong D_k(\ker \partial_{k-2})$. P_0 is projective, and there exists an isomorphism

$$P = P^1 \cong P^2 \oplus D_1(\ker \partial_1).$$

But then P^2 is a complex that satisfies the conditions of the lemma and is 0 in degree 0. Proceeding by induction we can construct maps

$$P^n \cong P^{n+1} \oplus D_n(\ker \partial_{n-2}),$$

giving us the result desired. ☐

Now consider a diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & X \\ {\scriptstyle i\,\wr}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

Since i is a cofibration (and thus dimensionwise a monomorphism) we have a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0$$

with P_n projective for all n. Notice, in addition, that since i is a weak equivalence P must be acyclic. But then we can write $B \cong A \oplus P$, and, since each $D_n(A)$ is projective (in the sense that for any epimorphism $X \to Y$ we can lift the map $\operatorname{Hom}(D_n(A), X) \to \operatorname{Hom}(D_n(A), Y)$) we can define a map $B \to X$ by letting it be g on the A factor and any lift on the P factor.

It remains to show MC4. Notice that the argument in section 2.2 is easy to generalize to chain complexes, with the two sets of maps being $\{0 \to D^n\}$ (for the factorization into acyclic cofibration-fibration) and $\{S^{n-1} \to D^n\}$ (for the factorization into cofibration-acyclic fibration). ☐

3 Simplicial Objects

An important example of model categories arises in the category of simplicial objects over a category $\mathcal{C}$.

3.1 Definitions

Homotopy groups are notoriously difficult to compute. How does one compute *all* maps from a sphere to a space? And when are two such maps homotopic? In topological spaces, this is a very difficult question. It turns out, however, that it is possible to create a combinatorial approximation to a topological space, based on the idea of simplicial complexes.

A simplicial complex is a collection of simplices glued together at the boundaries. All the structure that is imposed on the complex is that every n-simplex must have $n+1$ boundary simplices which are also included in the complex. In addition, there are certain consistency constraints on simplices that are the boundaries of n-simplices, as these must be glued together along $n-2$-simplices. The fact that these objects are simplices is irrelevant; we can simply define them to be sets and simply impose constraints on the boundary maps. This reasoning leads directly to the definition of a simplicial set:

Definition 3.1. A *simplicial set* is a collection $\{K_n\}_{n=0}^{\infty}$ of sets together with maps $d_i : K_n \to K_{n-1}$ and $s_i : K_n \to K_{n+1}$ for $0 \leq i \leq n$, satisfying the following conditions:

$$\begin{array}{rrl} d_i d_j = d_{j-1} d_i & d_i s_j = s_{j-1} d_i & \text{if } i < j \\ d_j s_j = d_{j+1} s_j = \mathrm{id} & s_i s_j = s_{j+1} s_i & \text{if } i \leq j \\ & d_i s_j = s_j d_{i-1} & \text{if } i > j+1 \end{array}$$

The maps d_i are called the *face maps* and the maps s_i are called the *degeneracy maps*.

It is not difficult to see that the conditions on the commutativity of the maps are the same as the *reverse* of the conditions on the commutativity of maps between ordered finite sets. Thus we have the following equivalent characterization of simplicial sets:

Lemma 3.2. *A simplicial set is a contravariant functor from the category* Δ^{op} *of finite ordered sets to* **Sets**.

This lemma motivates the following definition.

Definition 3.3. A *simplicial object* in a category $\mathcal{C}$ is a contravariant functor $\Delta^{\mathrm{op}} \to \mathcal{C}$. The category of such functors is denoted by $s\mathcal{C}$.

Now we will take a more detailed look at the category of simplicial sets.

3.2 Simplicial Sets

We first introduced the idea of simplicial sets as a combinatorial approximation to topological spaces. In order to better develop this analogy we first make some constructions about the structure of simplicial sets, and the basis of their equivalence with **Top**.

Consider a simplicial set $K \in s\mathbf{Sets}$. For each $n \geq 0$ it has a set K_n of the n-simplices of the simplicial set; the elements of the set K_0 are called the *vertices* of K. We call an element $k \in K_n$ *non-degenerate* if $k \neq s_i k'$ for any $0 \leq i \leq n$ and $k' \in K_{n-1}$. We write that k' is a *face* of k if $k' = d_i k$ for some i.

First we describe a simplicial map (natural transformation between the functors defining simplicial sets) combinatorially.

Definition 3.4. A *simplicial map* f between $K, L \in s\mathbf{Sets}$ is a collection of maps f_n $(n \geq 0)$ such that $f_n d_i = d_i f_{n+1}$ and $f_{n+1} s_i = s_i f_n$ for all i, n.

It turns out that for many general results about simplicial sets it suffices to consider only a very small class of simplicial objects, namely the standard n-simplices, their boundaries and horns.

Definition 3.5. The simplicial set Δ^n (the standard n-simplex) is the simplicial set where

$$K_i = \{(a_0, \ldots, a_i) \,|\, a_0 \le a_1 \le \cdots \le a_i,\ \{a_0, \ldots, a_i\} \subseteq \{0, \ldots, n\}\},$$

the simplicial complex Δ^n completed by degenerate simplices. The i-th face map removes a_i from a tuple; the i-th degeneracy map repeats a_i in a tuple.

The simplicial set $\partial\Delta^n$, the boundary of Δ^n, consists of the subcomplex of Δ^n where the subset condition above is replaced by strict containment. This will be the usual boundary of Δ^n, completed by degenerate simplices.

The simplicial set Λ^n_k is the largest subcomplex of $\partial\Delta^n$ where K_{n-1} does not contain the tuple $(0, 1, \ldots, k-1, k+1, \ldots, n)$. This is the k-th n-horn.

Recall that a singular n-simplex on a space X is a continuous map

$$\Delta_n = \{(t_0, \ldots, t_n) \,|\, t_i \ge 0,\ \sum_{i=0}^{n} t_i = 1\} \to X.$$

Notice that we have a functor Sing : **Top** $\to$ s**Sets**, which takes a topological space to the set of singular simplices on that space. We can define face and degeneracy maps on a simplex τ by defining

$$\delta_i : \Delta_{n-1} \to \Delta_n \qquad \delta_i(t_0, \ldots, t_{n-1}) = (t_0, \ldots, t_i, 0, t_{i+1}, \ldots, t_{n-1})$$

and

$$\sigma_i : \Delta_{n+1} \to \Delta_n \qquad s_i(t_0, \ldots, t_{n+1}) = (t_0, \ldots, t_i + t_{i+1}, \ldots, t_{n+1});$$

and writing

$$d_i\tau = \tau \circ \delta_i \qquad s_i\tau = \tau \circ \sigma_i.$$

Definition 3.6. Let $X \in s\mathbf{Sets}$. Define $\Delta \downarrow X$ to be the category of morphisms $\Delta^n \to X$ for $n \ge 0$ (with morphisms the obvious commutative triangles). Then we define the *geometric realization* functor $|\cdot| : s\mathbf{Sets} \to \mathbf{Top}$ by

$$|X| = \varinjlim_{\Delta\downarrow X} |\Delta^n|.$$

Lemma 3.7. $|\cdot|$ *is left adjoint to* Sing.

Proof. We follow the proof in [4]. Notice that

$$X \cong \varinjlim_{\Delta\downarrow X} \Delta^n,$$

as Δ is a small category (and therefore X is a colimit of representable functors). In addition, it is pretty clear that $|\Delta^n| = \Delta_n$. Therefore there are isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Top}}(|X|, Y) &\cong \varprojlim_{\Delta\downarrow X} \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, Y) \\
&\cong \varprojlim_{\Delta\downarrow X} \mathrm{Hom}_{s\mathbf{Sets}}(\Delta^n, \mathrm{Sing}Y) \\
&\cong \mathrm{Hom}_{s\mathbf{Sets}}(X, \mathrm{Sing}Y)
\end{aligned}$$

where the second line follows because $\mathrm{Hom}_{\mathbf{Top}}(\Delta_n, Y) = (\mathrm{Sing}Y)_n$. □

In order for simplicial sets to be useful as a combinatorial approximation to **Top** we want to construct a model category structure on simplicial sets that is equivalent to the model category structure on **Top**. In particular, we would like to develop a notion of homotopy groups of a simplicial set, and a notion of what it means for two maps to be homotopic.

Unlike in the topological space case, however, it turns out that it's easier to define a homotopy group first, and to allow the definition of simplicial homotopy to follow from the model category structure.

Definition 3.8. We call $n+1$ n-simplices

$$x_0, \dots, x_{k-1}, x_{k+1}, \dots, x_{n+1}$$

compatible if $d_i x_j = d_{j-1} x_i$ when $i < j$, $i, j \neq k$.

A simplicial set K is said to *satisfy the extension condition* or to be a *Kan complex* if, for every compatible collection of $n+1$ n-simplices $x_0, \dots, x_{k-1}, x_{k+1}, \dots, x_{n+1}$, there exists an $n+1$-simplex z such that $d_i z = x_i$ for $i \neq k$.

For example, any simplicial set in the image of Sing is a Kan complex. Notice that the extension condition is equivalent to the simplices being in the image of a simplicial map $\Lambda_k^{n+1} \to K$. Since $|\Lambda_k^{n+1}|$ is a deformation retract of Δ_{n+1} (it is just $n+1$ of the $n+2$ sides of the simplex) any continuous map $|\Lambda_k^{n+1}| \to X$ can be extended to a map $\Delta_{n+1} \to X$, which means in particular that Sing$X$ will be a Kan complex.

We now define the homotopy groups of a Kan complex K in terms of the extension condition.

Definition 3.9. Let K be a Kan complex and $x, x' \in K_n$. We write $x \sim x'$ if there exists $y \in K_{n+1}$ such that

$$d_n y = x \qquad d_{n+1} y = x' \qquad d_i y = s_{n-1} d_i x = s_{n-1} d_i x' \text{ if } 0 \leq i < n.$$

y is called a *homotopy* from x to x'.

Lemma 3.10. *$\sim$ is an equivalence relation on K_n.*

Proof. We follow the proof in [6]. Notice that $\sim$ is clearly reflexive, since, taking $y = s_n x$ we see that y is a homotopy from x to itself. It thus suffices to show that if $x \sim x'$ and $x \sim x''$ then $x' \sim x''$. Let y' be a homotopy from x to x', and let y'' be a homotopy from x to x''. Note that

$$d_i y' = s_{n-1} d_i x' = s_{n-1} d_i x = d_i y''$$

so we see that the $n+2$ simplices

$$d_0 s_n s_n x', \dots, d_{n-1} s_n s_n x', y', y''$$

satisfy the extension condition. Thus from the extension condition there exists a z such that these simplices are the boundary (except missing one simplex). But notice that $d_i d_{n+2} z = s_{n-1} d_i x'$ if $0 \leq i < n$ and $d_n d_{n+2} z = x'$ and $d_{n+1} d_{n+2} z = x''$. Thus $d_{n+2} z$ is a homotopy from x' to x'', so we see that $x' \sim x''$, as desired. ☐

We now define homotopy groups of a Kan complex. In topological spaces, we consider the n-th homotopy group to be the homotopy classes of images of the n-sphere in the space. Since an n-sphere can be considered to be an n-cell with a point attached, we define homotopy groups of a Kan complex to be equivalence classes of simplices whose entire boundary is contained in a point.

Definition 3.11. Any point $* \in K_0$ generates a subcomplex of K where the i-th dimension is $\{s_0 \cdots s_0 *\}$. We will write $*$ for the image of $* \in K_0$ in any dimension.

Then

$$\pi_n(K, *) = \{x \in K_n \mid d_i x = *, \ 0 \leq i \leq n\} / \sim .$$

$\pi_0(K, *)$ is called the set of path components of K.

Proposition 3.12. *Let $\alpha, \beta \in \pi_n(K, *)$, with $[x] = \alpha$, $[y] = \beta$. Then the $n+1$ n-simplices $*, \dots, *, x, \_, y$ are compatible, so there is a homotopy z from x to y. We define*

$$\alpha\beta = d_n z.$$

With this multiplication, $\pi_n(K, *)$ is a group for $n \geq 1$, and is abelian for $n \geq 2$.

Proof. We follow [6]. By an argument analogous to that of the proof that $\sim$ is an equivalence relation it is not difficult to show that multiplication is well-defined.

Notice that if we let $\beta = [*]$ then the definition of multiplication becomes exactly analogous to the definition that two simplices are homotopic. Thus $[*]$ is the identity.

We will construct a left inverse for x; a right inverse can be constructed analogously. Notice that the $n+1$ simplices $*, \dots, *, \_, x, y$ are compatible, so there is a z such that $d_i z = *$ for $i < n-1$, $d_n z = x$ and $d_{n+1} z = y$. Then by definition we have $[d_{n-1} z]\alpha = \beta$; letting $\beta = [*]$ shows that inverses exist.

Lastly we prove associativity. Pick $\alpha, \beta, \gamma \in \pi_n(K, *)$, with representatives x, y, z. Let $w_{n-1}, w_{n+1}, w_{n+2}$ be the simplices used for computing $\alpha\beta, \alpha\gamma, \beta\gamma$, respectively. Then by the extension condition we can choose a $u \in K_{n+2}$ to extend

$$*, \dots, *, w_{n-1}, w_{n+1}, w_{n+2}.$$

Then

$$(\alpha\beta)\gamma = [d_n w_{n-1}]\gamma = [d_{n-1} w_{n+1}]\gamma = [d_n w_{n+1}] = [d_n d_n u] = \alpha[d_n w_{n+2}] = \alpha(\beta\gamma).$$

It remains to show that $\pi_n(K, *)$ is abelian for $n \geq 2$. Let $w, x, y, z \in K_n$. This proof relies on three facts, which are proven analogously to the proof above:

1. Suppose that v satisfies

$$d_i v = * \; 0 \leq i < n-2 \qquad d_{n-2} v = w \quad d_{n-1} v = x \quad d_n v = y \quad d_{n+1} v = *.$$

   Then $[y][w] = [x]$.

2. Suppose that v satisfies

$$d_i v = * \; 0 \leq i < n-2 \qquad d_{n-2} v = w \quad d_{n-1} v = * \quad d_n v = y \quad d_{n+1} v = z.$$

   Then $[w][y] = [z]$.

3. Suppose that v satisfies

$$d_i v = * \; 0 \leq i < n-2 \qquad d_{n-2} v = w \quad d_{n-1} v = x \quad d_n v = y \quad d_{n+1} v = z.$$

   Then $[w]^{-1}[x][z] = [y]$.

Let $z = *$. Then $[w]^{-1}[x] = [y]$. Applying (1) to the v from (3) we find $[y] = [x][w]^{-1}$. Thus for all x $[x][w]^{-1} = [w]^{-1}[x]$. ☐

Now we have two ways of defining a model category structure on s**Sets**. The first of these would simply be to define it through the functor $|\cdot|$, defining a map to be a weak equivalence, fibration or cofibration if its image under $|\cdot|$ is. On the other hand, we can define the following model category structure on simplicial sets:

Definition 3.13. We define a map $f : X \to Y \in s\mathbf{Sets}$ to be a

- weak equivalence if it induces isomorphisms on the groups $\pi_n(\mathrm{Sing}|X|, v) \to \pi_n(\mathrm{Sing}|Y|, fv)$ and a bijection $\pi_0(\mathrm{Sing}|X|) \to \pi_0(\mathrm{Sing}|Y|)$,
- fibration if it has the RLP with respect to all inclusions $\Lambda^n_k \hookrightarrow \Delta^n$, and
- cofibration if it has the LLP with respect to all acyclic fibrations.

It turns out that these two definitions of the model category structure are equivalent, in that they yield equivalent homotopy categories. (For a proof of this, see [4], chapter I.)

Theorem 3.14. *The structure defined above is a model category structure on* s**Sets**. *In addition, the functors* Sing *and* $|\cdot|$ *induce an equivalence of categories* Ho **Top** *and* Ho s**Sets**.

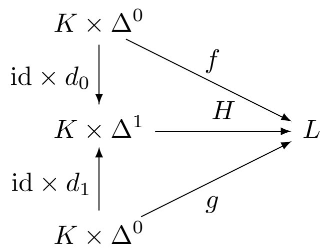
We omit the proof of this theorem because it would take us too far afield. For a proof of this, see [4].

However, we can also define simplicial homotopy analogously to the definition in **Top**: since Δ^1 is a simplicial set with $|\Delta^1| = I$ it makes sense to try and define homotopic maps through a construction similar to that of topological spaces.

Definition 3.15. Let $K, L \in s\mathbf{Sets}$. We define

$$(K \times L)_n = K_n \times L_n.$$

Now suppose that we have two maps $f, g : K \to L$. We say that f and g are *simplicially homotopic* if there is a map $H : K \times \Delta^1 \to L$ that makes the following diagram commute:



In this case we write $f \overset{\sim}{\to} g$. The map H is the *simplicial homotopy* between f and g.

Later, when we discuss simplicial categories, we will see that this notion is equivalent to that of a model category-theoretic homotopy, as $K \times \Delta^1$ will be a cylinder object for K.

3.2.1 A Couple of Results about Simplicial Groups

In this section we will prove a couple of results about simplicial groups that will be used later in the paper, when constructing the E^2 model category structure. We include them here because they are good, simple examples of computing with simplicial sets, and because they fit better into the theme of this section.

Lemma 3.16. *The underlying simplicial set of any simplicial group is fibrant.*

Proof. We follow the proof in [4]. To prove this we need to show that any simplicial group H satisfies the extension condition. We will prove this by induction on the number of compatible n-simplices.

First, notice that when we only have one simplex x (which is automatically compatible) the simplex $s_0 x$ has $d_0(s_0 x) = x$, so the simplicial group satisfies the extension condition for 1 simplex. Now suppose that for any m compatible simplices $(x_0, \ldots, x_{k-1}, x_\ell, \ldots, x_n)$ the simplicial group satisfies the extension condition (where $m = k + n - \ell + 1$). Consider $m + 1$ compatible simplices

$$(x_0, \ldots, x_{k-1}, x_{\ell-1}, x_\ell, \ldots, x_n).$$

Applying the induction hypothesis to all but $x_{\ell-1}$ we get a simplex $y \in H_{n+1}$. Notice that the m simplices $(1, \ldots, 1, x_{\ell-1} d_{\ell-1} y^{-1}, 1, \ldots, 1)$ (where the first 1's are spaces $0, \ldots, k-1$ and the last ones are $\ell, \ldots, n$) are compatible, and in addition $d_i(s_{\ell-2}(x_{\ell-1} d_{\ell-1} y^{-1}) y) = x_i$ for $i \leq k-1$ and $i \geq \ell - 1$. So the simplicial group H satisfies the extension condition for $m + 1$ simplices, completing the induction. □

This lemma easily yields the following corollary:

Corollary 3.17. *A map of simplicial groups is a fibration if and only if it is surjective.*

Proposition 3.18. **Ch** *is equivalent to* s**Ab**.

Proof. We follow [6]. This proof consists of on two steps. First we will show that there is an isomorphism between the homotopy of a simplicial abelian group and the homology of the groups considered as a chain complex. Second, we show that there is a pair of functors $\mathbf{Ch} \leftrightarrows s\mathbf{Ab}$ that compose to the identity and that map chain homotopies to simplicial homotopies (and vice versa).

Let $G \in s\mathbf{Ab}$. We write $A(G)$ for G considered as a chain complex with differential $\partial = \sum_i (-1)^i d_i$. We claim that $\pi_n(G) \to H_n(A(G))$ is an isomorphism for all n. Define the chain complex

$$N_n(G) = G_n \cap \ker d_0 \cap \cdots \cap \ker d_{n-1}.$$

The differential on this complex is $(-1)^n d_n$. We define a filtration of $A(G)$ by

$$F_p(G) = \{x \in A(G) \mid d_i x = 0 \; 0 \leq i < \min(n,p)\}.$$

Thus we see that $F_{-1}(G) = A(G)$ and $\bigcap_p F_p(G) = N(G)$. We will write $i_p : F_{p+1}(G) \hookrightarrow F_p(G)$ for the inclusion of F_{p+1} into F_p. We can define a map $f_p : F_p \to F_{p+1}$ by

$$f_p(x) = \begin{cases} x & \text{if } x \in F_p(G_n),\ n \leq p \\ x - s_p d_p x & \text{if } x \in F_p(G_n),\ n > p \end{cases}$$

Notice that $f_p i_p$ is the identity, and that $\partial f_p = f_p \partial$. Thus all we need to do now is to show that there is a chain homotopy between $i_p f_p$ and id. We define the homotopy t_P to be

$$t_p(x) = \begin{cases} 0 & \text{if } x \in F_p(G_n),\ n < p \\ (-1)^p s_p x & \text{if } x \in F_p(G_n),\ n \geq p. \end{cases}$$

A simple calculation shows that $\partial t_p(x) + t_p \partial(x) = x - i_p f_p(x)$, as desired.

Thus we see that $\pi_n(G) \to H_n(A(G))$ is an isomorphism. We will use this to construct the functors that will show the equivalence of $s\mathbf{Ab}$ and $\mathbf{Ch}$. In the previous paragraph we constructed a functor $N : s\mathbf{Ab} \to \mathbf{Ch}$. Now we will construct a functor $\Gamma : \mathbf{Ch} \to s\mathbf{Ab}$ and prove that it has the desired properties. We define $\Gamma(X)_n = X_n \oplus \bigoplus_{r-0}^{n-1} \sigma_{j_{n-r}} \cdots \sigma_{j_1} X_r$, (where $\sigma_{j_{n-r}} \cdots \sigma_{j_1} X_r$ is the abelian group composed of elements of the form $\sigma_{j_{n-r}} \cdots \sigma_{j_1} x$ for $x \in X_r$) for all sequences of the form $j_{n-r} > \cdots > j_1$. In particular, $\Gamma(X)_n$ consists of the nondegenerate part (X_n) and the degenerate part (everything else). We define d_i to be zero on X_n if $i \neq n$ and to be ∂ otherwise, and to be defined by the simplicial relations on the degenerate part. The degeneracy maps are defined in the obvious fashion.

Then we claim that N and Γ satisfy the desired properties. In order to see that $N\Gamma = 1_{\mathbf{Ch}}$ and $\Gamma N = 1_{s\mathbf{Ab}}$ notice that $A(G) = N(G) \oplus D(G)$, where $D(G)$ is the chain complex generated by the degenerate elements in $A(G)$. Notice that if we define $i = \cdots \circ i_n \circ \cdots i_0$ (this is well-defined in every dimension since it is eventually the identity on any dimension) and $f = \cdots \circ f_n \circ \cdots \circ f_0$, then fi is the identity on $N(G)$. Thus $A(G) = N(G) \oplus \ker f$, and it is clear that $\ker f = D(G)$.

Thus it remains to show the condition on homotopy. If we have a homotopy h between maps $f, g : G \to G' \in s\mathbf{Ab}$, we can define the map $s = \sum (-1)^i h_i$ on $A(G_n)$. Then s is a homotopy between $A(f)$ and $A(g)$ and it is clear that s induces a chain homotopy between $N(f)$ and $N(g)$ (as $N(G)$ is the quotient of $A(G)$ by $D(G)$). Similarly, let s be a chain homotopy between maps $f, g : X \to X' \in \mathbf{Ch}$. We can define a homotopy h between $\Gamma(f)$ and $\Gamma(g)$ by the following:

1. if $x \in X_n$ then $h_n(x) = s_n f(x) - s_n s\partial(x) - s(x)$, $h_{n-1}(x) = s_{n-1} f(x) - s_n s\partial(x)$, and $h_i(x) = s_i f(x)$ if $i \leq n-2$.

2. on $\sigma_{j_r} \cdots \sigma_{j_1} x$

$$h_i(\sigma_{j_r} \cdots \sigma_{j_1} x) = \begin{cases} \sigma_{j_r} h_{i-1}(\sigma_{j_{r-1}} \cdots \sigma_{j_1}) & \text{if } j_k \leq i-1 \\ \sigma_{j_r+1} h_i(\sigma_{j_{r-1}} \cdots \sigma_{j_1}) & \text{otherwise} \end{cases}$$

A simple computation shows that this is a homotopy, as desired. □

In particular, the proof of this theorem shows that the homotopy theory of simplicial abelian groups is equivalent to the homology of chain complexes.

3.3 Simplicial Categories

In a model category $\mathcal{C}$ we defined a notion of homotopy equivalence between two morphisms. Thus for any $X, Y \in \mathcal{C}$ we can construct a graph whose vertices are the elements of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ and whose edges are the homotopies between maps. We can therefore define a functor from the sets of morphisms between cofibrant-fibrant objects of a model category to $s\mathbf{Sets}$ that takes the hom-sets to simplicial sets. As a generalization of the we can try to define such a functor from $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to s\mathbf{Sets}$ that will, in the case $\mathcal{C}$ is a model category, encode this information. In this section, we will first look at categories that have such a functor in general, and then develop a notion of when such a functor is compatible with the model category structure on a category.

Definition 3.19. A category $\mathcal{C}$ is called a *simplicial category* if there exists a functor

$$\mathbf{Hom}(\cdot,\cdot) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to s\mathbf{Sets}$$

with the following properties:

1. $\mathbf{Hom}(A, B)_0 = \mathrm{Hom}_{\mathcal{C}}(A, B)$.
2. The functor $\mathbf{Hom}(A, \cdot) : \mathcal{C} \to s\mathbf{Sets}$ has a left adjoint $A \otimes \cdot : s\mathbf{Sets} \to \mathcal{C}$ such that there is a natural isomorphism
$$A \otimes (K \times L) \cong (A \otimes K) \otimes L.$$
3. The functor $\mathbf{Hom}(\cdot, B) : \mathcal{C}^{\mathrm{op}} \to s\mathbf{Sets}$ has a left adjoint $B^{\cdot} : s\mathbf{Sets} \to \mathcal{C}$ such that
$$\mathrm{Hom}_{s\mathbf{Sets}}(K, \mathbf{Hom}_{\mathcal{C}}(A, B)) \cong \mathrm{Hom}_{\mathcal{C}}(A, B^K).$$

In order for the category to be a simplicial model category it needs to be consistent with the notion of homotopy imposed by the model category structure. For example, we would like $\pi_0\mathbf{Hom}_{\mathcal{C}}(A, X) = [A, X]_{\mathcal{C}}$.

Definition 3.20 (Axiom SM7). A simplicial category $\mathcal{C}$ which is also a model category is a *simplicial model category* if, for all cofibrations $i : A \to B$ and fibrations $p : X \to Y$ we have that

$$\mathbf{Hom}_{\mathcal{C}}(B, X) \overset{(i^*, p_*)}{\longrightarrow} \mathbf{Hom}_{\mathcal{C}}(A, X) \times_{\mathbf{Hom}_{\mathcal{C}}(A,Y)} \mathbf{Hom}_{\mathcal{C}}(B, Y)$$

is a fibration of simplicial sets, which is acyclic if either i or p is acyclic.

In this section, we will first prove some statements about simplicial categories, culminating with a theorem about conditions on a category $\mathcal{C}$ that guarantee that $s\mathcal{C}$ will be a simplicial category. Then we will direct our attention to simplicial model categories to derive an equivalent condition to axiom SM7.

First, a couple of lemmas following from the definitions of a simplicial category.

Lemma 3.21.

1. *For a fixed $K \in s\mathbf{Sets}$ there exist adjoint functors*
$$K \otimes \cdot : \mathcal{C} \leftrightarrows \mathcal{C} : \cdot^K.$$
2. *For all $K, L \in s\mathbf{Sets}$, and for each $B \in \mathcal{C}$ there exists a natural isomorphism*
$$B^{K \times L} \cong (B^K)^L.$$
3. *For all $n \geq 0$,*
$$\mathbf{Hom}_{\mathcal{C}}(A, B)_n \cong \mathrm{Hom}_{\mathcal{C}}(A \otimes \Delta^n, B).$$

Proof. We follow [4].

1. Notice that for fixed $K \in s\mathbf{Sets}$, $A \in \mathcal{C}$ we know that $A \otimes K$ represents the functor

$$\mathrm{Hom}_{s\mathbf{Sets}}(K, \mathbf{Hom}_{\mathcal{C}}(A, \cdot)) : \mathcal{C} \to \mathbf{Sets}.$$

   Then if we have a map $f \in \mathrm{Hom}(A, A')$ we get a map on representing objects $A \otimes K \xrightarrow{f \otimes 1} A' \otimes K$, which gives us the functor $\cdot \times K$. We can obtain the functor $\cdot^K$ analogously. Adjointness follows because

$$\mathrm{Hom}_{\mathcal{C}}(A \otimes K, B) \cong \mathrm{Hom}_{s\mathbf{Sets}}(K, \mathbf{Hom}_{\mathcal{C}}(A, B)) \cong \mathrm{Hom}_{\mathcal{C}^{op}}(B^K, A) \cong \mathrm{Hom}_{\mathcal{C}}(A, B^K).$$

2. This follows from the adjointness proven in the first section and the fact that there is a natural isomorphism

$$(A \otimes K) \otimes L \cong A \otimes (K \times L).$$

3. Note that

$$\mathbf{Hom}_{\mathcal{C}}(A, B)_n \cong \mathrm{Hom}_{s\mathbf{Sets}}(\Delta^n, \mathbf{Hom}_{\mathcal{C}}(A, B)) \cong \mathrm{Hom}_{\mathcal{C}}(A \otimes \Delta^n, B)$$

   where the second part follows from the definition of $A \otimes \cdot$.

□

Notice that we can define a composition pairing on **Hom**-simplicial sets. Let $f \in \mathbf{Hom}_{\mathcal{C}}(A, B)$ and $g \in \mathbf{Hom}_{C}(B, C)$. In particular, this means that $f : A \otimes \Delta^n \to B$ and $g : B \otimes \Delta^n \to C$. Then we can define gf to be the composition

$$A \otimes \Delta^n \xrightarrow{1 \otimes \mathrm{diag}} A \otimes \Delta^n \otimes \Delta^n \xrightarrow{f \otimes 1} B \otimes \Delta^n \xrightarrow{g} C.$$

Notice that this is associative, reduces to ordinary composition in degree 0, and has a unit, in the sense that

$$\begin{array}{ccc} \mathrm{id} \times \mathbf{Hom}_{\mathcal{C}}(A, B) & = & \mathbf{Hom}_{\mathcal{C}}(A, B) \\ \downarrow & \nearrow & \\ \mathbf{Hom}_{\mathcal{C}}(A, A) \times \mathbf{Hom}_{\mathcal{C}}(A, B) & & \end{array}$$

commutes. From this construction we immediately get

Lemma 3.22. *In a simplicial category $\mathcal{C}$ we have the following adjointness isomorphisms:*

1. $\mathbf{Hom}_{s\mathbf{Sets}}(K, \mathbf{Hom}_{\mathcal{C}}(A, B)) \cong \mathbf{Hom}_{\mathcal{C}}(A \otimes K, B)$.
2. $\mathbf{Hom}_{s\mathbf{Sets}}(K, \mathbf{Hom}_{\mathcal{C}}(A, B)) \cong \mathbf{Hom}_{\mathcal{C}}(A, B^K)$.

This construction gives us a way to construct examples of simplicial categories.

Proposition 3.23. *Let $\mathcal{C}$ be a category. Suppose that there exists a functor*

$$\cdot \otimes \cdot : \mathcal{C} \times s\mathbf{Sets} \to \mathcal{C}$$

such that

1. *if $K \in s\mathbf{Sets}$, $\cdot \otimes K : \mathcal{C} \to \mathcal{C}$ has a right adjoint $\cdot^K$,*
2. *for a fixed A, $A \otimes \cdot : s\mathbf{Sets} \to \mathcal{C}$ commutes with all colimits and $A \otimes * \cong A$, and*
3. *there exists an isomorphism $A \otimes (K \times L) \cong (A \otimes K) \otimes L$, which is natural in A, K, L.*

Then $\mathcal{C}$ is a simplicial category with

$$\mathbf{Hom}_{\mathcal{C}}(A, B)_n = \mathrm{Hom}_{\mathcal{C}}(A \otimes \Delta^n, B).$$

In particular, this proposition says that if we have a definition of a tensor product that satisfies all of the conditions of a simplicial category, it is enough to define the simplicial structure. It follows that $s\mathbf{Sets}$ is a simplicial category (as it really ought to be) with the definition

$$A \otimes K = A \times K.$$

Proof. Notice that $\mathbf{Hom}_{\mathcal{C}}(A, B)_0 = \mathrm{Hom}_{\mathcal{C}}(A, B)$ because $A \otimes * \cong A$.

Now we prove that the second part of the definition. We can write K as the coequalizer

$$\coprod_q \Delta^{n_q} \rightrightarrows \coprod_p \Delta^{n_p} \to K.$$

Applying $A \otimes \cdot$, we get a coequalizer

$$\coprod_q A \otimes \Delta^{n_q} \rightrightarrows \coprod_p A \otimes \Delta^{n_p} \to A \otimes K.$$

But this means that we have an equalizer diagram

$$\mathrm{Hom}_{\mathcal{C}}(A \otimes K, B) \to \mathrm{Hom}_{\mathcal{C}}(A \otimes \left(\coprod_p \Delta^{n_p}\right), B) \rightrightarrows \mathrm{Hom}_{\mathcal{C}}(A \otimes \left(\coprod_q \Delta^{n_q}\right), B),$$

which this is equivalent to the equalizer diagram

$$\mathrm{Hom}_{s\mathbf{Sets}}(K, \mathbf{Hom}_{\mathcal{C}}(A, B)) \to \mathrm{Hom}_{s\mathbf{Sets}}(\coprod_p \Delta^{n_p}, \mathbf{Hom}_{\mathcal{C}}(A, B)) \rightrightarrows \mathrm{Hom}_{s\mathbf{Sets}}(\coprod_q \Delta^{n_q}, \mathbf{Hom}_{\mathcal{C}}(A, B)).$$

Thus **Hom** is right-adjoint to $A \otimes \cdot$, as desired.

The last part of the definition follows analogously to the second part, using the fact that $A \otimes \cdot$ has a right adjoint. □

Consider the conditions in the above lemma. The conditions only depend on the construction of the $\otimes$ functor. Notice that, as long as we can define a functor $\cdot \otimes \cdot$ on a category it will be a simplicial category. The graded structure of simplicial objects in $s\mathcal{C}$ are well-suited to this context.

Theorem 3.24. *Suppose that $\mathcal{C}$ is a category which is complete and cocomplete. Then $s\mathcal{C}$ is a simplicial category, with $\cdot \otimes \cdot$ defined by*

$$(A \otimes K)_n = \coprod_{k \in K_n} A_n$$

and face and degeneracy maps induced by the maps in K. Then we can define

$$\mathbf{Hom}_{s\mathcal{C}}(A, B)_n = \mathrm{Hom}_{s\mathcal{C}}(A \otimes \Delta^n, B).$$

Proof. We follow the proof in [4]. Since $*$ has only one element in each dimension, it is obvious that $A \otimes * \cong A$. Notice that from the above construction we automatically get a natural isomorphism

$$A \otimes (K \times L) \cong (A \otimes K) \otimes L.$$

Thus it suffices to show that, for $K \in s\mathbf{Sets}$, the functor $\cdot \otimes K$ has a right adjoint.

Define the functor $F_Y : \mathcal{C}^{\mathrm{op}} \to \mathbf{Sets}$ by $F_Y(A) = \mathrm{Hom}_{\mathcal{C}}(A, Y)$. Then the functor

$$A \longmapsto \mathbf{Hom}_{s\mathbf{Sets}}(K, F_Y(A))_n = \mathrm{Hom}_{\mathbf{Sets}}(K_n \times \Delta^n, \mathrm{Hom}_{s\mathcal{C}}(A, Y))$$

will be representable, as we can write $K \times \Delta^n$ as the coequalizer of

$$\coprod_q \Delta^{n_q} \rightrightarrows \coprod_p \Delta^{n_p} \to K \times \Delta^n$$

which implies that the representing object is determined by the equalizer diagram

$$\prod_q Y_{n_q} \leftleftarrows \prod_p Y_{n_p} \leftarrow (Y^K)_n.$$

Taking this construction for all n we get an object Y^K and a natural isomorphism of sets

$$\mathrm{Hom}_{s\mathcal{C}}(A, Y^K) \cong \mathrm{Hom}_{s\mathbf{Sets}}(K, \mathrm{Hom}_{s\mathcal{C}}(A, Y)),$$

which produces an equivalence of functors

$$F_{Y^K}(\cdot) \cong \mathrm{Hom}_{s\mathbf{Sets}}(K, F_Y(\cdot)).$$

From this we see that $\mathrm{Hom}_{s\mathcal{C}}(X, Y)$ is in one-to-one correspondence with the natural transformations $F_X \to F_Y$ (which we will denote by $\mathrm{Nat}(F_X, F_Y)$). Thus if $K \in s\mathbf{Sets}$ and $X \in s\mathcal{C}$ we can define

$$F_X \otimes K : \mathcal{C}^{\mathrm{op}} \to s\mathbf{Sets}$$

to be $(F_X \otimes K)(A) = F_X(A) \times K$.

We will now show that $\mathrm{Nat}(F_{X\otimes K}, F_Y) \cong \mathrm{Nat}(F_X \otimes K, F_Y)$. Consider a natural transformation $\Phi : F_X \otimes K \to F_Y$. Notice that

$$(F_X \otimes K)(X_n)_n = \prod_{k \in K_n} \mathrm{Hom}_{\mathcal{C}}(X_n, X_n).$$

Thus for all $k \in K_n$ there is a map $\Phi(1)_k : X_n \to X_n$. We can assemble them into a complex so that $f_n(X \otimes K)_n = \coprod_{k \in K_n} X_n \to Y_n$, so we get a simplicial map $f : X \otimes K \to Y$. Then $\Phi \mapsto f$ is the desired isomorphism.

Now notice that

$$\mathrm{Hom}_{s\mathcal{C}}(X, Y^K) \cong \mathrm{Nat}(F_X, F_{Y^K}) \cong \mathrm{Nat}(F_X, \mathrm{Hom}_{s\mathbf{Sets}}(K, F_Y)).$$

However, the set on the right-hand side is exactly equal to

$$\mathrm{Nat}(F_X \otimes K, F_Y) \cong \mathrm{Nat}(F_{X\otimes K}, F_Y) \cong \mathrm{Hom}_{s\mathcal{C}}(X \otimes K, Y).$$

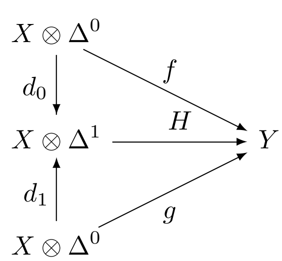
Thus there is a natural isomorphism

$$\mathrm{Hom}_{s\mathcal{C}}(X, Y^K) \cong \mathrm{Hom}_{s\mathcal{C}}(X \otimes K, Y)$$

as desired. □

Note, in addition, that in a simplicial category we can define homotopic maps even if the category is not a model category by analogy from the definition of simplicial homotopy in $s\mathbf{Sets}$.

Definition 3.25. Two maps $f, g : X \to Y \in s\mathcal{C}$ are *simplicially homotopic* if there exists a map $H : X \otimes \Delta^1 \to Y$ that makes the following diagram commute:



The map H is called a *simplicial homotopy* between f and g.

In later sections we will be constructing simplicial model category structures on $s\mathcal{C}$, so as a prelude to that we will now develop some theory of when a category is a simplicial model category. In particular, we will show that instead of checking all pairs of a cofibration and a fibration in SM7 it is possible to check a statement for a class of maps containing only a subset of the cofibrations.

For the rest of this section, unless otherwise stated, $\mathcal{C}$ will refer to a simplicial model category.

Lemma 3.26. *Let $q : X \to Y$ be a fibration. Then for cofibrant B*

$$q_* : \mathbf{Hom}_{\mathcal{C}}(B, X) \to \mathbf{Hom}_{\mathcal{C}}(B, Y)$$

is a fibration. Analogously, if $j : A \to B$ is a cofibration and X is fibrant then

$$j^* : \mathbf{Hom}_{\mathcal{C}}(B, X) \to \mathbf{Hom}_{\mathcal{C}}(A, X)$$

is a fibration.

Proof. Axiom SM7 reduces to the first statement if we let $A = \varnothing$. SM7 reduces to the second statement if we let $B = *$. □

It turns out, in fact, that axiom SM7 encodes some of the properties of the model category structure on $\mathcal{C}$.

Lemma 3.27. *Axiom SM7 implies axiom MC3.*

Proof. We follow [4]. Consider a diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

where one of i and p is acyclic. Notice that this diagram is a vertex in

$$\mathbf{Hom}_{\mathcal{C}}(A, X) \times_{\mathbf{Hom}_{\mathcal{C}}(A,Y)} \mathbf{Hom}_{\mathcal{C}}(B, Y),$$

the maps $A \to X$ and $B \to Y$ that agree on A. The diagrams that have lifts $B \to X$ are the ones that are in the image of the map

$$\mathbf{Hom}_{\mathcal{C}}(B, X) \xrightarrow{(i^*, p_*)} \mathbf{Hom}_{\mathcal{C}}(A, X) \otimes_{\mathbf{Hom}_{\mathcal{C}}(A,Y)} \mathbf{Hom}_{\mathcal{C}}(B, Y).$$

We know that this map is an acyclic fibration since one of i and p is acyclic. However, the acyclic fibrations in $s\mathbf{Sets}$ are surjective, so a lift exists. □

Lemma 3.28. *Let $j : K \to L \in s\mathbf{Sets}$. If $A \in \mathcal{C}$ is cofibrant then* $\mathrm{id} \otimes j : A \otimes K \to A \otimes L$ *is a cofibration; if $X \in \mathcal{C}$ is fibrant then* $\mathrm{id}^j : X^L \to X^K$ *is a fibration. If j is acyclic, then* $\mathrm{id} \otimes j$ *and* id^j *are, also.*

Proof. We follow [4]. For the first claim, we will show that $\mathrm{id} \otimes j$ has the LLP with respect to the acyclic fibrations. Suppose that we have a diagram

$$\begin{array}{ccc} A \otimes K & \longrightarrow & X \\ \big\downarrow & & \big\downarrow\wr \\ A \otimes L & \longrightarrow & Y \end{array}$$

Using the fact that $A \otimes \cdot$ is adjoint to $\mathbf{Hom}_{\mathcal{C}}(A, \cdot)$ we get the diagram

$$\begin{array}{ccc} K & \longrightarrow & \mathbf{Hom}_{\mathcal{C}}(A, X) \\ \big\downarrow & \overset{\exists}{\nearrow} & \big\downarrow\wr \\ L & \longrightarrow & \mathbf{Hom}_{\mathcal{C}}(A, Y) \end{array}$$

where the right-hand map is an acyclic fibration by axiom SM7. Notice that if j is acyclic we could use an analogous argument to show that $\mathrm{id} \otimes j$ is an acyclic cofibration.

The second part is proven analogously. □

Consider the simplicial set Δ^1. Its geometric realization is that of the unit interval, so that in **Top** we use it to construct cylinder objects. It it reasonable, therefore, to expect that we could use Δ^1 to construct cylinder objects.

Proposition 3.29. *Letting $q : \Delta^1 \to *$ be the map to the terminal object. Then for all $A \in \mathcal{C}$ the map*

$$\mathrm{id} \otimes q : A \otimes \Delta^1 \to A \otimes *$$

is a weak equivalence,

$$A \amalg A \stackrel{d_0 \amalg d_1}{\longrightarrow} A \otimes \Delta^1$$

is a cofibration, and

$$A \amalg A \longrightarrow A \otimes \Delta^1 \longrightarrow A \cong A \otimes *$$

is the map $\mathrm{id} \amalg \mathrm{id}$.

In particular, it follows from this proposition that $A \otimes \Delta^1$ is a cylinder object for A. (And, as we mentioned earlier, this means that for a simplicial set K, $K \times \Delta^1$ is a cylinder object for K, implying that simplicial homotopy is equivalent to the model category-theoretic homotopy.) Notice, too, that this means that in a simplicial model category homotopic maps are simplicially homotopic, and vice versa, so that the definition of simplicial homotopy is consistent with the model category definition of homotopy.

Proof. We follow [4]. Notice that the map $\Delta^0 \to \Delta^1$ is an acyclic cofibration. Then by proposition 3.28 we know that

$$A \otimes \Delta^0 \rightarrow\rightarrow A \otimes \Delta^1$$

is a weak equivalence. Since the identity is also a weak equivalence, by MC1 we know that

$$\mathrm{id} \otimes q$$

must also be one.

The map $A \amalg A \to A \otimes \Delta^1$ is a cofibration using proposition 3.28, since $A \amalg A \cong A \otimes \partial\Delta^1$.

Lastly we need to show that $(1 \otimes q)d_1 = (1 \otimes q)d_0 = \mathrm{id}$. However, this is clear as this is $A \otimes \cdot$ applied to the composition $\partial\Delta^1 \to \Delta^1 \to *$, which is the fold map. ☐

We now state two dual statements that will be used to prove two restatements of Axiom SM7.

Proposition 3.30. *Let $\mathcal{C}$ be a simplicial category and a model category.*

1. *Axiom SM7 holds if and only if for all cofibrations $i : K \to L \in s$**Sets** and $j : A \to B \in \mathcal{C}$, the map*

$$(A \otimes L) \amalg_{A \otimes K} (B \otimes K) \longrightarrow B \otimes L$$

is a cofibration, which is acyclic if i or j is.

2. *Axiom SM7 holds if and only if, for all fibrations $q : K \to L \in s$**Sets** and $p : X \to Y \in \mathcal{C}$, the map*

$$X^L \longrightarrow X^K \otimes_{Y^K} Y^L$$

is a fibration, which is acyclic if q or p is.

Proof. We follow the proof in [4]. Notice that these statements are duals of one another, so we will only prove the first; the second is shown analogously.

First, suppose that Axiom SM7 holds. Let $i : K \hookrightarrow L \in s$**Sets** and $j : A \to B \in \mathcal{C}$. We want to show that

$$(A \otimes L) \amalg_{A \otimes K} (B \otimes K) \longrightarrow B \otimes L$$

is a cofibration, and is acyclic if i or j is. Let $p : X \to Y$ be a fibration. A diagram

$$\begin{array}{ccc} (A \otimes L) \amalg_{A \otimes K} (B \otimes K) & \longrightarrow & X \\ \downarrow & & \downarrow \\ B \otimes L & \longrightarrow & Y \end{array}$$

implies, by adjointness, a diagram

$$\begin{array}{ccc} K & \longrightarrow & \mathbf{Hom}_{\mathcal{C}}(B,X) \\ {\scriptstyle i}\big\downarrow & \overset{\exists}{\nearrow} & \big\downarrow \\ L & \longrightarrow & \mathbf{Hom}_{\mathcal{C}}(B,Y) \times_{\mathbf{Hom}_{\mathcal{C}}(A,Y)} \mathbf{Hom}_{\mathcal{C}}(B,Y) \end{array}$$

in which a lift exists if i or j is acyclic, or if p is acyclic, so there is a lift in the first diagram, as well. In the first case, we see that $(A \otimes L) \amalg_{A \otimes K} (B \otimes K) \to B \otimes L$ must have been an acyclic cofibration (since p could be any fibration); in the second, it will be a cofibration (as then p would have to be acyclic).

The other direction is done analogously, with the lift existing in the first diagram implying a lift in the second one. □

Using the above proposition we can construct equivalent statements of Axiom SM7:

Theorem 3.31. *Let $i : K \hookrightarrow L \in s\mathbf{Sets}$, and let $p : X \twoheadrightarrow Y \in s\mathcal{C}$. Then the following are equivalent:*

1. *Axiom SM7*
2. *(Axiom SM7a)*
$$X^{\Delta^n} \longrightarrow X^{\partial\Delta^n} \times_{Y^{\Delta^n}} Y^{\partial\Delta^n}$$
is a fibration (and is acyclic if p is), and
$$X^{\Delta^1} \longrightarrow X^e \times_{Y^e} Y^{\Delta^1}$$
is an acyclic cofibration for $e = 0, 1$.
3. *(Axiom SM7b)*
$$(A \otimes \Delta^n) \amalg_{A \otimes \partial\Delta^n} (B \otimes \partial\Delta^n) \longrightarrow B \otimes \Delta^n$$
is a cofibration (and is acyclic if i is), and that
$$(A \otimes \Delta^1) \amalg_{A \otimes e} (B \otimes e) \longrightarrow B \otimes \Delta^1$$
is an acyclic cofibration for $e = 0, 1$.

Proof. We follow [4]. We will prove that (1) is equivalent to (3) using the first part of proposition 3.30; the proof that (1) is equivalent to (2) will follow analogously from the second part of the proposition.

First, notice that (1) clearly implies (3) by proposition 3.30. It suffices to show that (3) implies the condition in proposition 3.30. Consider any cofibration $i : K \to L \in s\mathbf{Sets}$. Since K is a colimit of a diagram involving only Δ^n's, the condition

$$(A \otimes \Delta^n) \amalg_{A \otimes \partial\Delta^n} (B \otimes \partial\Delta^n) \to B \otimes \Delta^n$$

is a cofibration if j is implies that

$$(A \otimes L) \amalg_{A \otimes K} (B \otimes K) \to B \otimes L$$

is a cofibration that is trivial if j is.

The second condition follows because fibrations in simplicial sets can be defined as the maps having the RLP with respect to the maps

$$(\Delta^1 \otimes \partial\Delta^n) \amalg (\{e\} \otimes \Delta^n) \to \Delta^1 \times \Delta^n$$

for $e = 0, 1$.

□

4 The Reedy Model Category Structure

In the previous section we saw that by taking advantage of the simplicial construction of $s\mathcal{C}$, $s\mathcal{C}$ becomes a simplicial category if $\mathcal{C}$ is complete and cocomplete. The natural next question to ask is whether $s\mathcal{C}$ can inherit some kind of model category structure from $\mathcal{C}$; ideally, this model category structure would also make $s\mathcal{C}$ into a simplicial model category. The Reedy model category structure, which can be defined on $s\mathcal{C}$ for any model category $\mathcal{C}$ makes $s\mathcal{C}$ into a model category. Also, although it does not make $s\mathcal{C}$ into a simplicial model category under the standard construction, if $\mathcal{C}$ itself is a simplicial model category then the Reedy structure inherits the simplicial structure from $\mathcal{C}$ that makes $s\mathcal{C}$ into a simplicial model category.

4.1 Definition of the Reedy Structure

First, a couple of definitions.

Definition 4.1. Let $\Delta_{[n]}$ be the category of finite ordered sets of cardinality at most $n+1$. Define $s_n\mathcal{C}$ to be the category of contravariant functors $\Delta_{[n]}^{\mathrm{op}} \to \mathcal{C}$. The functor $R_n : s\mathcal{C} \to s_n\mathcal{C}$ composes a functor $X : \Delta^{\mathrm{op}} \to \mathcal{C}$ with the inclusion $\Delta_{[n]} \to \Delta$.

We define the *skeleton* functor sk_n to be the left adjoint to R_n, and the *coskeleton* functor ck_n to be the right adjoint to R_n. We define the *n-th latching object* L_nX to be $\mathrm{sk}_{n-1}(R_{n-1}X)_n$, and the *n-th matching object* M_nX to be $\mathrm{ck}_{n-1}(R_{n-1}X)_n$.

We first develop alternate, more computationally simple, definitions of the latching and matching objects.

Proposition 4.2.

$$L_nX = \varinjlim_{\substack{[n]\to[k]\\ k<n}} X_k \qquad M_nX = \varprojlim_{\substack{[j]\to[n]\\ j<n}} X_j.$$

Proof. First, consider the case for the latching object. sk_{n-1} is the left adjoint to the restriction functor, so that

$$\mathrm{Hom}(X, R_{n-1}Y) \cong \mathrm{Hom}(\mathrm{sk}_{n-1}X, Y).$$

For $k < n$, this simply means that $\mathrm{sk}_{n-1}(X)_k = X_k$. In the n-th dimension this means that $\mathrm{sk}_{n-1}(X)_n$ is the object such that any map into a Y that commutes with degeneracy maps on dimensions less than n must commute with degeneracy maps out of the n-th dimension. In particular, any map $X_k \to Y_k$ for all $k < n$ must extend to a map $\mathrm{sk}_{n-1}(X)_n \to Y_n$ that commutes with the degeneracy maps. But this exactly means that $\mathrm{sk}_{n-1}(X)_n$ is the colimit of all of the degeneracy maps out of X_n, subject to the constraints that define degeneracy maps. In particular, it means that if we take $\mathcal{D}$ to be the category whose objects are maps in Δ^n that will map to degeneracy maps, and whose morphisms are the commutative triangles between them, L_nX will be the colimit of the obvious functor $\mathcal{D} \to \mathcal{C}$. This is exactly the definition given above.

The matching object case is done analogously. □

Notice that there are natural maps $L_nX \to X_n$ and $X_n \to M_nX$, as there are maps $X_k \to X_n$ (for the former) and $X_n \to X_k$ (for the latter) with the face and degeneracy maps commuting properly.

We now use these constructions to define a model category structure on $s\mathcal{C}$.

Definition 4.3. The *Reedy model category structure* on $s\mathcal{C}$ defines a map $X \to Y \in s\mathcal{C}$ to be a

- weak equivalence if for all $n \geq 0$ $X_n \to Y_n$ is a weak equivalence.
- cofibration if for all $n \geq 0$, $X_n \amalg_{L_nX} L_nY \to Y_n$ is a cofibration.
- fibration if for all $n \geq 0$, $X_n \to Y_n \times_{M_nY} M_nX$ is a fibration.

It will be useful to have a more direct characterization for acyclic Reedy fibrations and cofibrations. It turns out that such a characterization, completely analogous to the fibration and cofibration definitions, exists.

Proposition 4.4.

1. *A Reedy fibration is acyclic if and only if*

$$X_n \longrightarrow M_n X \times_{M_n Y} Y_n$$

is an acyclic fibration for all $n \geq 0$*.*

2. *A Reedy cofibration is acyclic if and only if*

$$X_n \amalg_{L_n X} L_n Y \longrightarrow Y_n$$

is an acyclic cofibration for all $n \geq 0$*.*

Before we prove this, we will introduce generalized matching and latching objects, which will make it possible to prove properties of Reedy fibrations and cofibrations by induction. For greater detail, see [4].

Proposition 4.5. *Let* $K \in s\mathbf{Sets}$*. Then the functor* $\cdot \otimes K : \mathcal{C} \to s\mathcal{C}$ *has a right adjoint* M_K*. For a fixed* X*, the map* $K \mapsto M_K X$ *is a functor which has a left adjoint (and therefore preserves limits).*

Proof. We know that

$$\mathrm{Hom}_{s\mathcal{C}}(Z \otimes K, X) \cong \mathrm{Hom}_{s\mathcal{C}}(Z, X^K).$$

Thus, as for constant $Z \in s\mathcal{C}$, $\mathrm{Hom}_{s\mathcal{C}}(Z, Y) \cong \mathrm{Hom}_{\mathcal{C}}(Z, Y_0)$, we can define $M_K X = (X^K)_0$ and satisfy the adjointness condition.

In addition,

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(Z, M_K X) &\cong \mathrm{Hom}_{s\mathcal{C}}(Z \otimes K, X) \\ &\cong \mathrm{Hom}_{s\mathbf{Sets}}(K, \mathbf{Hom}_{s\mathcal{C}}(Z, X)) = \mathrm{Hom}_{s\mathbf{Sets}^{\mathrm{op}}}(\mathbf{Hom}_{s\mathcal{C}}(Z, X), K), \end{aligned}$$

so the left adjoint exists. $\square$

Proposition 4.6. *Let* J *be a small category with a functor* $F : J \to \Delta$*. The functor* $L_J : s\mathcal{C} \to \mathcal{C}$ *defined by*

$$L_J X = \varinjlim_{J} X \circ F$$

has a right adjoint.

Proof. Notice that $\mathrm{Hom}_{\mathcal{C}}(L_J X, Z) \cong \mathrm{Hom}_{\mathcal{C}^J}(X \circ F, Z)$ (where we consider Z to be a constant diagram). But the right-hand side is equal to $\mathrm{Hom}_{s\mathcal{C}}(X, F^! Z)$, where $F^!$ is the Kan extension functor, so $L_J X$ does have a right adjoint. $\square$

Definition 4.7. The *generalized matching object* of X with respect to a simplicial set K is $M_K X$. The *generalized latching object* of X with respect to a small category J is $L_J X$.

The definitions of $M_K X$ and $L_J X$ quickly yield the following lemma:

Lemma 4.8.

$$M_{\Delta^n} X \cong L_{\Delta^n} X \cong X_n.$$

Now we are ready to prove the characterization of acyclic Reedy fibrations and cofibrations.

Proof of proposition 4.4. We follow [4].

1. We define $\Delta^{n,k}$ to be the set $d^0\Delta^n \cup \cdots \cup d^k\Delta^n$. Clearly, $\Delta^{n,-1} = \varnothing$ and $\Delta^{n,n} = \partial\Delta^n$. Notice that there is a pushout diagram of sets

$$\begin{array}{ccc} \Delta^{n-1,k} & \longrightarrow & \Delta^{n-1} \\ \downarrow & & \downarrow \\ \Delta^{n,k} & \longrightarrow & \Delta^{n,k+1} \end{array}$$

Taking matching objects, we get a pullback square (where we denote $M_{\Delta^{n,k}}$ by $M_{n,k}$)

$$\begin{array}{ccc} M_{n,k+1} & \longrightarrow & X_{n-1} \\ \downarrow & & \downarrow \\ M_{n,k}X & \longrightarrow & M_{n-1,k}X \end{array}$$

which in turn yields the pullback square

$$\begin{array}{ccc} Y_n \times_{M_{n,k+1}Y} M_{n,k+1}X & \longrightarrow & X_{n-1} \\ \downarrow & & \downarrow \\ Y_n \times_{M_{n,k}Y} M_{n,k}X & \longrightarrow & Y_{n-1} \times_{M_{n-1,k}Y} M_{n-1,k}X \end{array} \tag{1}$$

Now suppose that $f : X \to Y$ is an acyclic Reedy fibration. We will prove by induction that $X_n \to Y_n \times_{M_nY} M_nX$ is an acyclic fibration. When $n = 0$, this map is just $X_0 \to Y_0$; this is a fibration because f is a fibration, and a weak equivalence because f is a weak equivalence. Now suppose that for all $-1 \leq k \leq n-1$

$$X_{n-1} \to Y_{n-1} \times_{M_{n-1,k}Y} M_{n-1,k}X$$

is an acyclic fibration. Then the right-hand side vertical map in (1) is an acyclic fibration, which means that for all $-1 \leq k \leq n-1$ we know that

$$Y_n \times_{M_{n,k+1}} M_{n,k+1}X \longrightarrow Y_n \times_{M_{n,k}Y} M_{n,k}X$$

is an acyclic fibration. In particular, we get the following diagram:

$$X_n \twoheadrightarrow Y_n \times_{M_{n,n}Y} M_{n,n}X \overset{\sim}{\twoheadrightarrow} Y_n \times_{M_{n,n-1}Y} M_{n,n-1}X \overset{\sim}{\twoheadrightarrow} \cdots \overset{\sim}{\twoheadrightarrow} Y_n$$

The composition of all of these maps is f_n, which we know is a weak equivalence because f is a weak equivalence. Thus by the two-out-of-three axiom we see that $X_n \to Y_n \times_{M_nY} M_nX$ must be an acyclic fibration, as desired. This completes the induction.

Notice that this argument works in the reverse direction, as well. We use induction to get the analogous diagram whose composition is f_n, except that in this case every fibration will be a weak equivalence, so that the composition will also be a weak equivalence. Thus f will be a weak equivalence and therefore an acyclic Reedy fibration.

2. This part is done analogously to the first part, except that partial latching objects are used instead of partial matching objects.

□

Theorem 4.9. *For any model category $\mathcal{C}$, the above definitions give a model category structure on $s\mathcal{C}$.*

Proof. We follow [9]. We need to check the four model category axioms.

First, we know that a weak equivalence is simply a weak equivalence in every dimension. Since a map in $s\mathcal{C}$ is simply a collection of maps in $\mathcal{C}$ that are composed dimensionwise, MC1 follows immediately from the model category structure on $\mathcal{C}$.

Checking that a retract of a weak equivalence, fibration, or cofibration is another such map is similarly easy. For example, for the cofibration case it is a simple diagram chase to show that if $f : X \to Y$ is a retract of $g : X' \to Y'$, which is a cofibration, then for each $n \geq 0$ $X_n \amalg_{L_nX} L_nY \to Y_n$ is a retract of $X'_n \amalg_{L_nX'} L_nY' \to Y'_n$, which shows that f is also a cofibration. The other two cases are done similarly.

Now we will show that this satisfies MC3. Suppose that we have a diagram like this:

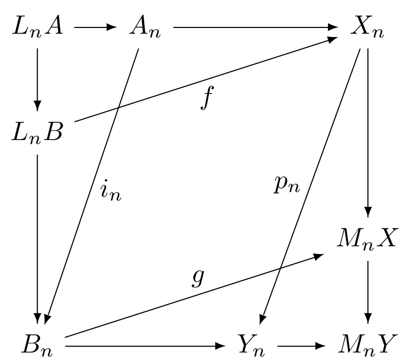
$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

where either i or p is acyclic. We will construct the lift degree-wise by induction. Restricting the above diagram to degree 0, and noting that the definitions of (acyclic) fibrations and cofibrations means that in the diagram

$$\begin{array}{ccc} A_0 & \longrightarrow & X_0 \\ {\scriptstyle i_0}\big\downarrow & & \big\downarrow{\scriptstyle p_0} \\ B_0 & \longrightarrow & Y_0 \end{array}$$

i_0 and p_0 are a cofibration and a fibration, respectively, and one of them is acyclic. Thus a lift $B_0 \to X_0$ exists.

Now suppose that a lift $\ell : B \to X$ exists up to degree $n-1$. By induction, for each map $\varphi : [n] \to [k] \in \Delta$ (for $k < n$) we have a map $B_k \to X_n$ defined by $X(\varphi) \circ \ell$. Therefore there is a map $f : L_n B \to X_n$. Similarly, for each map $[k] \to [n] \in \Delta$ (for $k < n$) we have a map $B_n \to X_k$ given by $\ell \circ B(\varphi)$. Therefore there is a map $g : B_n \to M_n X$. Using these definitions, we have the following commutative diagram:



where the maps $L_n B \to B_n$ and $X_n \to M_n X$ are the canonical maps. By taking the pushouts of the upper-left and lower-right corners of the diagram, we get the following:

$$\begin{array}{ccc} L_n B \amalg_{L_n A} A_n & \longrightarrow & X_n \\ {\scriptstyle i_*}\big\downarrow & & \big\downarrow{\scriptstyle p_*} \\ B_n & \longrightarrow & Y_n \times_{M_n Y} M_n X \end{array}$$

where i_* and p_* are a cofibration and a fibration, respectively, and one of them is acyclic (as these maps are exactly the maps in the conditions for i and p to be cofibrations/fibrations). Thus we see that a lift $B_n \to X_n$ exists that makes this diagram commute. In particular, from its construction we see that it forms a component of the lift $B \to X$ so that it now works up to degree n. Continuing, we see that we can make a lift $B \to X$ that makes the original diagram commute, as desired.

Lastly we need to show MC4. We first construct the acyclic cofibration - fibration factorization. Consider a map $X \to Y \in s\mathcal{C}$. We construct the factorization inductively.

When $n = 0$ we simply factor $X_0 \to Y_0$ into

$$X_0 \hookrightarrow^{\sim} Z(0) \twoheadrightarrow Y_0$$

Now suppose that we have a factorization

$$R_{n-1}X \hookrightarrow^{\sim} Z(n-1) \twoheadrightarrow R_{n-1}Y$$

(where in this diagram the cofibrations and fibrations mean that they satisfy the cofibration/fibration condition up to dimension $n-1$). Let $r_n : s_n\mathcal{C} \to s_{n-1}\mathcal{C}$ be the restriction functor. Then if we let r^* be the left adjoint to r_n and $r^!$ be the right adjoint to r_n we see that

$$r^* R_{n-1}X = R_n \mathrm{sk}_{n-1} X \qquad \text{and} \qquad r^! R_{n-1}X = R_n \mathrm{ck}_{n-1} X,$$

since the image of r_n in dimensions less than n will be the same as the image of R_n in dimensions less than n. Now consider the morphism $r_n r^* Z(n-1) \to Z(n-1)$. By adjointness it gives us a map $r^* Z(n-1) \to r^! Z(n-1)$ so that the following diagram commutes:

$$\begin{array}{ccccccc} r^* R_{n-1}X & \longrightarrow & r^* Z(n-1) & \longrightarrow & r^! Z(n-1) & \longrightarrow & r^! R_{n-1}Y \\ \downarrow & & & & & & \uparrow \\ R_n X & & & \longrightarrow & & & R_n Y \end{array}$$

Looking in just dimension n, we obtain

$$\begin{array}{ccccccc} L_n X & \longrightarrow & L_n Z(n-1) & \longrightarrow & M_n Z(n-1) & \longrightarrow & M_n Y \\ \downarrow & & & & & & \uparrow \\ X_n & & & \longrightarrow & & & Y_n \end{array}$$

Note that, since $L_n X$ and $M_n Y$ only depend on $R_{n-1}X$ and $R_{n-1}Y$, respectively, from our original factorization we get a map $L_n Z(n-1) \to L_n Y \to Y_n$ and a map $X_n \to M_n X \to M_n Z(n-1)$ that make the diagram commute. In particular, this means that in the above diagram we get a map in $\mathcal{C}$ from the pushout of the left-hand side to the pullback of the right-hand side,

$$X_n \amalg_{L_n X} L_n Z(n-1) \longrightarrow M_n Z(n-1) \times_{M_n Y} Y_n$$

Factor this map as

$$X_n \amalg_{L_n X} L_n Z(n-1) \overset{\sim}{\hookrightarrow} Z' \twoheadrightarrow M_n Z(n-1) \times_{M_n Y} Y_n$$

Define $Z(n)$ by $Z(n)_k = Z(n-1)_k$ for $k < n$, and $Z(n)_n = Z'$. Notice that the only thing we need to check is that in the map

$$R_n X \overset{\sim}{\twoheadrightarrow} Z(n) \twoheadrightarrow R_n Y$$

the face and degeneracy maps between the n-th and $n-1$-st dimensions work properly. (The cofibration/fibration conditions work from the definition of Z'.) However, from the same argument as in MC3, we see that because the maps $L_n Z \to Z' \to M_n Z$ work correctly, and because the face and degeneracy maps work correctly with relation to X_n and Y_n, this will in fact be a morphism in $s_n\mathcal{C}$.

Thus if we define $Z_n = Z(n)_n$ we will have a factorization

$$X \overset{\sim}{\hookrightarrow} Z \twoheadrightarrow Y$$

Notice that the cofibration-acyclic fibration factorization proof works completely analogously, with only the change that each factorization in $\mathcal{C}$ must be a factorization into a cofibration followed by an acyclic fibration. So we are done. □

4.2 Example: $s\mathbf{Ch}_R$ and Double Chain Complexes

We construct the model category structure on $s\mathbf{Ch}_R$. Since this category is equivalent to double chain complexes over R (as we can see by normalizing the simplicial groups in each dimension) the equivalence gives us a model category structure on double chain complexes. When discussing simplicial chain complexes we will picture a simplicial chain complex as a lattice in the first quadrant, where every column is a chain complex and the i-th row represents the i-th dimension of every chain complex in the simplicial chain complex. Thus an element of $s\mathbf{Ch}_R$ is a lattice of modules, where every row forms a simplicial module over R and every column forms a chain complex over R.

4.2.1 $s\mathbf{Ch}_R$

We will first develop some results about latching and matching objects in this category.

Proposition 4.10. *Let $X \in s\mathbf{Ch}_R$. We define*

$$\widehat{X}_n = \bigoplus_{i=0}^{n-1} s_i X_{n-1},$$

the module of degenerate simplices in X_n. Then $L_n X = \widehat{X}_n$.

Proof. We know that $L_n X = \mathrm{sk}_{n-1}(X)_n$. The sk_{n-1} functor is the left adjoint to the restriction functor from $s\mathbf{Ch}_R$ to $s_{n-1}\mathbf{Ch}_R$. In particular, notice that $\mathrm{sk}_{n-1}(X)$ merely removes all nondegenerate simplices from the set X_k for $k > n-1$. Thus in the n-th dimension it will have exactly all of the degenerate simplices, which are exactly the images of X_{n-1} in X. Thus $L_n X = \widehat{X}_n$, as desired. □

Proposition 4.11. *Let $X \in s\mathbf{Ch}_R$. Let $P\Delta^n$ be the category whose objects are the elements of the sets in the simplicial set $\partial\Delta^n$, and whose morphisms are the maps d_i and s_i. We define* $\mathrm{Maps}(\partial\Delta^n, X)$ *to be* $\mathrm{Hom}(R_{n-1}(\partial\Delta^n), R_{n-1}X)$*. Then*

$$M_n X = \mathrm{Maps}(\partial\Delta^n, X).$$

Proof. We know that $M_n X = \mathrm{ck}_{n-1}(X)_n$. ck_{n-1} is the right adjoint to the restriction functor; in particular, this means that $\mathrm{ck}_{n-1}(X)$ is the smallest module which is consistent with any simplicial chain complex that can map into the restriction to the lowest n dimensions of X. In the n-th dimension, this means that we must have enough objects so that they are consistent with $\partial\Delta^n$'s that are mapping into X from simplices in the other complex. However, this is exactly the definition above: we need an object for every possible map of a $\partial\Delta^n$ into X. □

We now consider the Reedy model category structure on $s\mathbf{Ch}_R$. First, the weak equivalences are clearly dimensionwise weak equivalences; thus they are the maps that are isomorphisms in vertical homology. Now consider the fibrations. We know that a map $f : X \to Y$ is a fibration if the map $X_n \to M_n X \times_{M_n Y} Y_n$ is a fibration in $\mathbf{Ch}_R$ for each $n > 0$. Notice that since fibrations are epimorphisms we can consider each chain complex dimensionwise. In particular, we can look at the simplicial groups X'_k and Y'_k that consist only of the k-th dimension elements of each chain complex. In order for f to be a fibration we need $(X'_k)_n \to M_n(X'_k) \times_{M_n Y'_k} (Y'_k)_n$ to be an epimorphism for all n and $k > 0$.

$M_n X'_k = \mathrm{Maps}(\partial\Delta^n, X'_k)$ and $M_n Y'_k = \mathrm{Maps}(\partial\Delta^n, Y'_k)$. We will write the element of $M_n X'_k$ that comes from an element x of X'_k by $\widetilde{x}$. In particular, we know that

$$M_n X'_k \times_{M_n Y'_k} (Y'_k)_n = \{(y, g) \mid g \in \mathrm{Maps}(\partial\Delta^n, X'_k), \widetilde{y} = f_n g\}.$$

The map that we are considering is the pullback of two maps: f_n and $\widetilde{\cdot}$. Thus in particular the image in $M_n X'_k \times_{M_n Y'_k} (Y'_k)_n$ is going to be of the form

$$\{(y, \widetilde{x}) \mid \widetilde{y} = f_n \widetilde{x}\}.$$

In order for this to be a fibration it needs to be an epimorphism. In particular, every $\varphi \in \mathrm{Maps}(\partial\Delta^n, X'_k)$ s.t. there is a y s.t. $\widetilde{y} = f_n\varphi$ must have some x such that $\widetilde{x} = \varphi$. Thus all cycles in homology which are boundaries in $(Y'_k)_n$ and are in the image of f_n must also be boundaries in $(X'_k)_n$. Then we have a lifting in every diagram of simplicial sets

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & X'_k \\ \big\downarrow & & \big\downarrow f_n \\ \Delta^n & \longrightarrow & Y'_k \end{array}$$

implying that f_n is an acyclic fibration of simplicial sets. In particular, we see that a map f is a fibration in $s\mathbf{Ch}_R$ if it is an acyclic fibration of simplicial sets on each "row". Since a fibration of simplicial groups is simply a surjective map, the fibrations are the surjective maps that are isomorphisms in horizontal homotopy in all but the first row.

4.2.2 Double Chain Complexes

Now we will translate the above model category structure to the category of double chain complexes, $\mathbf{dCh}_R$. A double chain complex X is a set modules indexed by pairs (m, n) with $m, n \geq 0$ with maps $d^{\leftarrow}_{m,n} : X_{m,n} \to X_{m-1,n}$ and $d^{\downarrow}_{m,n} : X_{m,n} \to X_{m,n-1}$ wherever these are well-defined, with the property that

$$d^{\leftarrow}_{m,n-1} d^{\downarrow}_{m,n} = d^{\downarrow}_{m-1,n} d^{\leftarrow}_{m,n}.$$

The functor $s\mathbf{Ch}_R$
$rightarrow \mathbf{dCh}_R$ acts by normalizing each row.

After the normalization, the class of weak equivalences does not change: these are the weak equivalences in vertical homology. The fibrations were acyclic fibrations on each row. As we saw in the proof of proposition 3.18, the homotopy groups of a simplicial abelian group map to homology groups in the chain complex. Thus we see that the condition for a map being a Reedy fibration in $\mathbf{dCh}_R$ is that it is an acyclic fibration in each row: it needs to be a surjective isomorphism in homology in each row.

Thus we get the following Reedy model category structure on $\mathbf{dCh}_R$. A map is

- a weak equivalence if it is an isomorphism in vertical homology,
- a fibration if it is surjective and an isomorphism in horizontal homology above the first row, and
- a cofibration if it has the left lifting property with respect to the acyclic fibrations.

4.3 Simplicial Structure

We now consider the question of a simplicial structure on $s\mathcal{C}$ that would make it into a simplicial model category with the associated Reedy model category structure.

First, suppose that $\mathcal{C}$ is in fact a simplicial model category. Then we can construct a simplicial structure on $s\mathcal{C}$ by simply "inheriting" the structure of $\mathcal{C}$: we define

$$(X \widehat{\otimes} K)_n = X_k \widehat{\otimes} K \qquad (X^K)_n = X_n^K \qquad \mathbf{Hom}_{s\mathcal{C}}(X, Y)_n = \mathrm{Hom}_{s\mathcal{C}}(X \otimes \Delta^n, Y),$$

where $\widehat{\otimes}$ is the the product that we are defining, distinguished from the $\otimes$ that is implied by theorem 3.24 because the category is simplicial objects over $\mathcal{C}$.

Proposition 4.12. *The above structure makes $s\mathcal{C}$ into a simplicial model category.*

Proof. We follow [4]. We want to show that for any fibration q,

$$X^{\Delta^n} \longrightarrow X^{\partial\Delta^n} \times_{Y^{\partial\Delta^n}} Y_n$$

is a fibration that is acyclic if q is, and also that

$$X^{\Delta^1} \longrightarrow X^e \times_{Y^e} Y^{\Delta^1}$$

is an acyclic fibration for $e = 0, 1$.

Let $Z = X^{\partial\Delta^n} \times_{Y^{\partial\Delta^n}} Y_n$. A map $X \to Z$ is a fibration if, for all n, $X_n \to M_n X \times_{M_n Z} Z_n$ is a fibration. This map is a fibration if q is, and is acyclic if q is. Thus we want to show that

$$X_m \to M_m X \times_{M_m Z} Z_m$$

is a fibration for all m. Expanding Z, we see that we want to show that

$$X_m^{\Delta^n} \to M_m(X)^{\Delta^n} \times_{M_m(X)^{\partial\Delta^n} \times_{M_m(Y)^{\partial\Delta^n}} M_m Y_n} (X^{\partial\Delta^n} \times_{Y^{\partial\Delta^n}} Y_n)_m$$

is a fibration for all m.

To do this it suffices to show that $M_n(Y^K) = M_n(Y)^K$, as the rest follows from the fact that this will be a pullback of fibrations. This is true because we have that for any $A \in \mathcal{C}$

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(A, M_n(Y^K)) &\cong \mathrm{Hom}_{s\mathcal{C}}(A \otimes \partial\Delta^n, Y^K) \cong \mathrm{Hom}_{s\mathcal{C}}((A \otimes \partial\Delta^n)\widehat{\otimes} K, Y) \\ &\cong \mathrm{Hom}_{s\mathcal{C}}((A\widehat{\otimes} K) \otimes \partial\Delta^n, Y) \cong \mathrm{Hom}_{s\mathcal{C}}(A\widehat{\otimes} K, M_n Y) \\ &\cong \mathrm{Hom}_{\mathcal{C}}(A, (M_n Y)^K),\end{aligned}$$

where the second line follows because

$$((A \otimes \partial\Delta^n)\widehat{\otimes} K)_m \cong \left(\coprod_{i=1}^{\binom{n}{m}} A\right) \otimes K \cong \coprod_{i=1}^{\binom{n}{m}} A \otimes K \cong (A\widehat{\otimes} K) \otimes \partial\Delta^n.$$

□

This is called the *internal* simplicial model category structure on $s\mathcal{C}$.

Notice, however, that the standard simplicial structure on $s\mathcal{C}$ does not turn $s\mathcal{C}$ into a simplicial model category with the Reedy model category structure. In particular, consider the map $\mathrm{id}\otimes d_0 : Z\otimes\Delta^0 \to Z\otimes\Delta^1$. For this to be an acyclic cofibration we need

$$(Z \otimes \Delta^0)_n \amalg_{L_n(Z\otimes\Delta^0)} L_n(Z \otimes \Delta^1) \to (Z \otimes \Delta^1)_n$$

to be an acyclic cofibration. In the case $n = 0$ this is

$$Z \to (Z \amalg Z).$$

If Z is cofibrant this will be a cofibration but not, in general, an acyclic one.

However, we do have the following proposition:

Proposition 4.13. *Let $j : K \to L \in s\mathbf{Sets}$ be a cofibration.*

1. *If $i : A \to B \in s\mathcal{C}$ is a Reedy cofibration then*

$$(X \otimes L) \amalg_{X\otimes K} (Y \otimes K) \longrightarrow Y \otimes L$$

is a Reedy cofibration which is acyclic if i is acyclic.

2. *If $p : X \to Y \in s\mathcal{C}$ is a Reedy fibration then*

$$X^L \longrightarrow X^K \times_{Y^K} Y^L$$

if a Reedy fibration which is acyclic if p is acyclic.

Proof. First, notice that these two statements are adjoint to one another. We will present the proof of the first; the second is proven analogously.

Notice that as $L_n K$ is the degenerate simplices in K_n (by the analogous argument to 4.10), we can easily see that

$$L_n(X \otimes K) = \coprod_{k\in L_n K} L_n X.$$

Thus we can compute that

$$(X \otimes L \amalg_{X\otimes K} Y \otimes K)_n = \left(\coprod_{\ell\in L_n} X_n\right) \amalg_{\coprod_{k\in K_n} X_n} \left(\coprod_{k\in K_n} Y_n\right) = \coprod_{\ell\in L_n\setminus K_n} X_n \amalg \coprod_{k\in K_n} Y_n$$

and

$$L_n(X \otimes L \amalg_{X\otimes K} Y \otimes K) = \left(\coprod_{\ell\in L_n L} L_n X\right) \amalg_{\coprod_{k\in L_n K} L_n X} \left(\coprod_{k\in L_n K} L_n Y\right) = \coprod_{\ell\in L_n L\setminus L_n K} L_n X \amalg \coprod_{x\in L_n K} L_n Y.$$

Thus in particular we see that the condition for the map $X \otimes L \amalg_{X\otimes K} Y \otimes K \hookrightarrow Y \otimes L$ to be an (acyclic) cofibration is equivalent the condition for f to be an (acyclic) cofibration. Thus we see that if f is an (acyclic) cofibration, then this map will be, too, as desired. □

5 The E^2 Model Category Structure

In the previous section, we constructed a model category structure on $s\mathcal{C}$ that was related to the model category structure on $\mathcal{C}$, but with one major flaw: with the standard simplicial structure on $s\mathcal{C}$ it did not impose a simplicial model category structure on $s\mathcal{C}$. To solve this problem we construct another model category structure on $s\mathcal{C}$ that is consistent with the simplicial structure; this structure is the E^2 model catgory structure, first introduced by Dwyer, Kan, and Stover in [2]. Unfortunately this structure is not as general as the Reedy model category structure, and requires several extra conditions on $\mathcal{C}$.

5.1 Definition of the E^2 Structure

Definition 5.1. We define a *pointed* model category $\mathcal{C}_*$ to be a model category in which the initial and terminal objects are equal.

In this section, we require $\mathcal{C}_*$ to be pointed, complete, cocomplete, and have every object fibrant.

Recall that in topological spaces, a weak equivalence is a map that is an isomorphism for all homotopy groups, meaning the homotopy classes of maps from spheres into the space. In constructing the E^2 model category structure we first construct an analogue of spheres that we will base our model category structure on.

Definition 5.2. A *cofibrant cogrouplike object* is a cofibrant object M, together with a homotopy class of maps $M \to M \amalg M$, such that this class imposes a group structure on $[M, Y]$ for any $Y \in \mathcal{C}_*$.

The n-th suspension object $\Sigma^n M$ of M is defined as follows. $\Sigma^0 M = M$. Given $\Sigma^n M$ and a factorization of $\Sigma^n M \to *$ into $\Sigma^n M \hookrightarrow C\Sigma^n M \xrightarrow{\sim} *$ we define $\Sigma^{n+1} M$ through the following pushout:

$$\begin{array}{ccc} \Sigma^n M & \hookrightarrow & C\Sigma^n M \\ \downarrow & & \downarrow \\ C\Sigma^n M & \hookrightarrow & \Sigma^{n+1} M \end{array}$$

Notice that in the case of topological spaces, if we let $M = S^1$ then $\Sigma^n M = S^{n+1}$ and $C\Sigma^n M = D^{n+2}$, the cone on $\Sigma^n M$. The following lemma easily follows from the definition; it is an important way of constructing infinitely many cofibrant cogrouplike objects.

Lemma 5.3. *If L is cofibrant then ΣL is a cofibrant cogrouplike object.*

Corollary 5.4. *If M is a cofibrant cogrouplike object, then so is $\Sigma^j M$ for all $j \geq 0$.*

Definition 5.5. The E^2 model category structure on a category $s\mathcal{C}_*$ is given by the following. A map $X \to Y$ in $s\mathcal{C}_*$ is a

- weak equivalence if for each $j \geq 0$ the induced map of simplicial groups $[\Sigma^j M, X] \to [\Sigma^j M, Y]$ is a weak equivalence of simplicial groups (i.e. it is a weak equivalence of the underlying simplicial sets). (The simplicial groups are formed by taking, for each n, $[\Sigma^j M, X_n] \in \mathcal{C}_*$ and then using the induced maps $X_n \to X_{n-1}$ and $X_n \to X_{n+1}$ to construct these into a simplicial group.)

- cofibration if it is a retract of an M-free map. A map $X \to Y$ is M-free if for each $n \geq 0$ there exists a cofibrant $Z_n \in \mathcal{C}_*$ and an index multiset $I_n \subset \mathbb{Z}$ such that $Z_n \xrightarrow{\sim} \amalg_{i \in I_n} \Sigma^i M$ and a map $Z_n \to Y_n$ such that
$$(X_n \amalg_{L_n X} L_n Y) \amalg Z_n \to Y_n$$
is an acyclic cofibration.

- fibration if it is a Reedy fibration and for every $j \geq 0$ the map $[\Sigma^j M, X] \to [\Sigma^j M, Y]$ is a fibration of simplicial groups (i.e. is a fibration of the underlying simplicial sets).

Although this definition is not obviously related to the Reedy model category structure, there are several important connections:

Proposition 5.6.

1. *The set of E^2 weak equivalences contains the Reedy weak equivalences.*
2. *The set of Reedy cofibrations contains the set of E^2 cofibrations. The set of E^2 acyclic cofibrations contains the set of acyclic Reedy cofibrations.*
3. *The set of E^2 fibrant objects contains the set of Reedy fibrant objects. The set of acyclic E^2 fibrations contains the set of acyclic Reedy fibrations.*

Proof.

1. Let $f : X \to Y$ be a Reedy weak equivalence, so f_n is a weak equivalence for all n. By definition, a weak equivalence is something that induces isomorphisms on homotopy groups in $\mathcal{C}$. Thus when we have a map of simplicial groups

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & [\Sigma^j M, X_{n+1}] & \longrightarrow & [\Sigma^j M, X_n] & \longrightarrow & [\Sigma^j M, X_{n-1}] & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f_{n+1}\circ} & & \downarrow{\scriptstyle f_n\circ} & & \downarrow{\scriptstyle f_{n-1}\circ} & & \\ \cdots & \longrightarrow & [\Sigma^j M, Y_{n+1}] & \longrightarrow & [\Sigma^j M, Y_n] & \longrightarrow & [\Sigma^j M, Y_{n-1}] & \longrightarrow & \cdots \end{array}$$

each of the vertical maps is an isomorphism. Since each vertical map is an isomorphism, we must have an isomorphism in simplicial homotopy for all n, so that we see that f is also an E^2 weak equivalence, as desired.

2. Let $i : X \to Y$ be an E^2 cofibration; we will show that it is also a Reedy cofibration. Note that it suffices to show that any M-free map is a Reedy cofibration, since Reedy cofibrations are closed under retracts. Since $X \to Y$ is an M-free map that means that there exist Z_n such that h is an acyclic cofibration.

$$\begin{array}{ccccc} \varnothing & \hookrightarrow & Z_n & \xrightarrow{\sim} & \coprod_{i \in I_n} \Sigma^i M \\ \downarrow & & \downarrow & & \\ X_n \amalg_{L_n X} L_n Y & \overset{g}{\hookrightarrow} & (X_n \amalg_{L_n X} L_n Y) \amalg Z_n & & \\ & \searrow\!\!{\scriptstyle (\text{dashed})} & & \overset{h}{\underset{\sim}{\hookrightarrow}} & Y_n \end{array}$$

In order for i to be a Reedy cofibration we need to show that the dashed map is a Reedy cofibration. Notice that g is a cofibration as it is the pushout of a cofibration; thus the dashed map is the composition of two cofibrations, and is therefore a cofibration. Thus i is a Reedy cofibration, as desired.

Now consider an Reedy acyclic cofibration $f : X \to Y$; we want to show that it is an E^2 acyclic cofibration. Note that since we already showed that any Reedy weak equivalence is an E^2 weak equivalence, it suffices to show that this is an E^2 cofibration. Let $Z_n = \varnothing$ (which is cofibrant, as any isomorphism is a cofibration). Then we have the following diagram:

$$\begin{array}{ccccc} \varnothing & \hookrightarrow & \varnothing & & \\ \downarrow & & \downarrow & & \\ X_n \amalg_{L_n X} L_n Y & \hookrightarrow & X_n \amalg_{L_n X} L_n Y & \overset{\sim}{\hookrightarrow} & Y_n \end{array}$$

where the map to Y_n is an acyclic cofibration by the definition of a Reedy acyclic cofibration. Thus f is an E^2 cofibration, and therefore an E^2 acyclic cofibration, as desired.

3. Suppose that we have a Reedy fibrant object X. In order for it to be E^2 fibrant it suffices to show that $[\Sigma^j M, X] \to [\Sigma^j M, *]$ is a fibration of simplicial groups for all j. Since $[\Sigma^j M, *]$ is a trivial group for

all j all that is necessary is to show that $[\Sigma^j M, X]$ is a fibrant simplicial group. However, all simplicial groups are fibrant, so X is also E^2 fibrant.

Lastly, we want to show that a Reedy acyclic fibration is also an E^2 acyclic fibration. Once again, we know that any Reedy weak equivalence is an E^2 weak equivalence, so it suffices to show that any Reedy acyclic fibration is an E^2 fibration. But in particular we know that any Reedy weak equivalence is an isomorphism on $[\Sigma^j M, X] \to [\Sigma^j M, Y]$, so we see that it will be a fibration of simplicial groups. Thus any Reedy acyclic fibration is an E^2 acyclic fibration, as desired.

□

Before we go on to the proof that this structure induces a simplicial model category structure on $s\mathcal{C}$ we will prove a statement about simplicially homotopic maps.

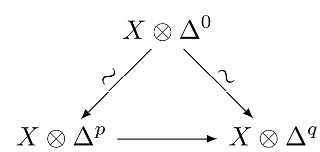
Proposition 5.7. *Suppose that $f, g : X \to Y$ are in the same connected component in $\mathbf{Hom}(X, Y)$. Then if f is an E^2 equivalence then so is g.*

Proof. It suffices to show this for simplicially homotopic maps. A simplicial homotopy between maps f and g can be viewed as a set of combinatorial data relating f_n and g_n. In particular, when we apply the functor $[\Sigma^j M, \cdot]$ to the maps the combinatorial data still holds. This means that the induced maps $f^*, g^* : [\Sigma^j M, X] \to [\Sigma^j M, Y]$ are also simplicially homotopic. But since simplicial sets are a model category this means that f^* and g^* are homotopic. Since f^* induces an isomorphism of simplicial groups, so does g^*, and therefore g must also be an E^2 weak equivalence. □

Corollary 5.8. *For any map $\Delta^p \to \Delta^q \in s\mathbf{Sets}$ and any $X \in s\mathcal{C}$, the induced map $X \otimes \Delta^p \to X \otimes \Delta^q \in s\mathcal{C}$ is an E^2 weak equivalence.*

Proof. Notice that if $K \to L$ is a simplicial homotopy equivalence in $s\mathbf{Sets}$ then the induced map $X \otimes K \to X \otimes L$ must also be a simplicial homotopy equivalence, as the combinatorial data stored in the homotopy equivalence is preserved after applying the functor $X \otimes \cdot$. Therefore $X \otimes K \to X \otimes L$ is an E^2 weak equivalence.

We know that $\Delta^0 \to \Delta^p$ is a simplicial homotopy equivalence. Then in the triangle



the two diagonal maps are E^2 weak equivalences. But then by MC1 we know that $X \otimes \Delta^p \to X \otimes \Delta^q$ must also be an E^2 weak equivalence, as desired. □

Now we are going to prove the main result of this section: that the model category structure defined above is indeed a model category structure, and that this structure is compatible with the simplicial structure on $s\mathcal{C}$.

First, a few lemmas that will be used in the proof of the model category structure. In particular, we will first prove two lemmas constructing factorizations of arbitrary maps in $s\mathcal{C}$.

Lemma 5.9. *We can factor any map into an acyclic E^2 cofibration that has the left lifting property with respect to all E^2 fibrations followed by a map which induces a fibration of simplicial groups.*

Proof. We follow [2]. Consider the map $\varphi_j : \Sigma^j M \otimes \Delta^0 \to \Sigma^j M \otimes \Delta^n$, induced by $\Delta^0 \to \Delta^n$, the inclusion of the 0-th vertex into the n-simplex. We let

$$P(j,n) = C\Sigma^j M \amalg_{\varphi_j} \Sigma^j M \otimes \Delta^n.$$

Then for any $Y \in s\mathcal{C}_*$ we can define

$$PY = \coprod_{(j,n,a,b)} P(j,n)$$

where $a : C\Sigma^j M \to Y_0$ and $b : \Sigma^j M \to Y_n$ are maps in $\mathcal{C}_*$ such that the inclusion of $\Sigma^j M$ into $C\Sigma^j M$ followed by a is equal to b restricted to the 0-th vertex.

Consider the map $* \to PY$; we will show that this map is an acyclic E^2 cofibration that has the left lifting property with respect to all E^2 fibrations. Note that it suffices to show all of these for each $P(j,n)$, since PY is simply a coproduct of all $P(j,n)$. $P(j,n)$ is defined as the pushout of $\Sigma^j M \otimes \Delta^n \xleftarrow{\sim} \Sigma^j M \otimes \Delta^0 \hookrightarrow \mathcal{C}\Sigma^j M \otimes \Delta^0$. Note that the map $\Sigma^j M \otimes \Delta^n \to P(j,n)$ will be an E^2 cofibration, as it is the pushout of an E^2 cofibration. Since $\Sigma^j M \otimes \Delta^n$ is cofibrant we see that $P(j,n)$ must also be cofibrant. The map $C\Sigma^j M \otimes \Delta^0 \to P(j,n)$ is a simplicial homotopy equivalence, as it is the cobase change of the simplicial homotopy equivalence $\Sigma^j M \otimes \Delta^0 \to \Sigma^j m \otimes \Delta^n$. In order to show that this map has the LLP with respect to the E^2 fibrations it suffices to show that the maps $* \to \Sigma^j M \otimes \Delta^n$ and $* \to C\Sigma^j M \otimes \Delta^0$ have this property. Let $X \to Y$ be an E^2 fibration. The following two diagrams show this,

$$\begin{array}{ccc} * & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \Sigma^j M \otimes \Lambda^n_k & \exists & \\ \downarrow & & \\ \Sigma^j M \otimes \Delta^n & \longrightarrow & Y \end{array} \qquad \begin{array}{ccc} \Sigma^j M \otimes \Delta^n & & \\ \downarrow & \searrow & \\ C\Sigma^j M \otimes \Delta^0 & \longrightarrow & X \\ \downarrow & \exists \nearrow & \downarrow \\ * & \longrightarrow & Y \end{array}$$

as the liftings that are obtained in the diagrams using the maps $\Sigma^j M \otimes \Lambda^n_k \to \Sigma^j M \otimes \Delta^n$ (where the diagram on the right is simply the case $n = 0$) clearly make the desired squares commute. Thus $* \to PY$ is a map as desired.

Now consider the map $\varphi : PY \to Y$ given by a and b on each of the $P(j,n)$ in the coproduct. This map induces, for each $j \geq 0$, a fibration of simplicial groups $[\Sigma^j M, PY] \to [\Sigma^j M, Y]$. Consider any map $f : \Sigma^j M, \to Y_m$. This map is the index is in the index of some copy of $P(j,m)$, so there is a map $\Sigma^j M \to PY$ that is mapped onto f. Analogously, for any map $C\Sigma^j M \to Y$ there also exists a lift $C\Sigma^j M \to PY$. Thus the map $[\Sigma^j M, PY] \to [\Sigma^j M, Y]$ is onto, so it is a fibration of simplicial groups (by corollary 3.17).

Now we can factor the map $* \to Y$ into an acyclic E^2 cofibration that has the left-lifting property with respect to all E^2 fibrations, followed by a map which induces a fibration on all simplicial groups. Thus if we have a map $f : X \to Y \in s\mathcal{C}_*$ then

$$\begin{array}{ccccc} * & \overset{\sim}{\hookrightarrow} & PY & & \\ \downarrow & & \downarrow & & \\ X & \overset{g}{\underset{\sim}{\hookrightarrow}} & X \amalg PY & \xrightarrow{f \amalg \varphi} & Y \end{array}$$

where g is an acyclic E^2 cofibration because it is the pushout of an acyclic E^2 cofibration (and therefore also has the LLP with respect to all E^2 fibrations), and $f \amalg \varphi$ induces a fibration of simplicial groups $[\Sigma^j M, X \amalg PY] \to [\Sigma^j M, Y]$, as desired. □

Lemma 5.10. *We can factor any map $X \to Y \in s\mathcal{C}_*$ as an E^2 cofibration $X \to \widetilde{Y}$ followed by a map $\widetilde{Y} \to Y$ that induces an acyclic fibration on the simplicial groups $[\Sigma^j M, \widetilde{Y}] \to [\Sigma^j M, Y]$.*

Proof. We follow [2]. We will first prove the above statement for constant simplicial complexes. Then we will use a diagonal argument to show that we can extend this to all simplicial objects.

In this section, we write τ_j for the inclusion of $\Sigma^j M$ in $C\Sigma^j M$. We construct a functor $V : \mathcal{C} \to \mathcal{C}$ by defining VX to be the pushout

$$\begin{array}{ccc} \coprod_{j\geq 0} \coprod_{h \in \mathrm{Hom}(C\Sigma^j M, X)} (\Sigma^j M)_{h\tau_j} & \longrightarrow & \coprod_{j\geq 0} \coprod_{h \in \mathrm{Hom}(C\Sigma^j M, X)} (C\Sigma^j M)_h \\ \downarrow & & \downarrow \\ \coprod_{j\geq 0} \coprod_{f \in \mathrm{Hom}(\Sigma^j M, X)} (\Sigma^j M)_f & \longrightarrow & VX \end{array}$$

Now consider a map $A \to B$ between constant simplicial objects; we will factor this map as $A \hookrightarrow \widehat{W} \xrightarrow{\sim} B$, where W_* is a non-constant simplicial object.

We will now define a functor $W : (A \downarrow \mathcal{C}) \to (A \downarrow \mathcal{C})$. For a map $A \to X$, define $WX = A \amalg VX$. Let $\epsilon : WX \to X$ be the natural map that sends a component $(\Sigma^j M)_f$ of VX into X by f, $(C\Sigma^j M)_h$ into X by h, and A into X by the map $A \to X$. Let $\beta : WX \to W^2X$ be the natural map that sends $(\Sigma^j M)_f$ into the copy of $\Sigma^j M$ that is indexed by the inclusion $\Sigma^j M \xrightarrow{f} V^2X$, $(C\Sigma^j M)_h$ into the copy of $C\Sigma^j M$ that is indexed by the inclusion $C\Sigma^j M \xrightarrow{h} V^2X$, and A into itself. Then (W, ϵ, β) forms a cotriple. Thus we can define a simplicial object $\widehat{W}$ by $\widehat{W}_n = W^{n+1}B$; we claim that $A \to \widehat{W}$ is an E^2 cofibration, and that $\widehat{W} \to B$ is an E^2 weak equivalence.

First we look at the map $A \to \widehat{W}$. Consider VX. We can select a factor in the coproduct forming this by choosing, for each map $f \in \mathrm{Hom}(\Sigma^j M, X)$ for which there exists an $h \in \mathrm{Hom}(C\Sigma^j M, X)$ with $h\tau = f$ exactly one $h \in \mathrm{Hom}(C\Sigma^j M, X)$ that satisfies this equation. Then this subcomplex will be weakly equivalent to $*$, while the quotient of VX by this subcomplex will be weakly equivalent to a coproduct of $\Sigma^j M$'s. Since VW_{n-1} has the homotopy type of a coproduct of $\Sigma^j M$'s, we can let $Z_n = V\widehat{W}_{n-1}$ in the definition of an E^2 cofibration. We need to show $A \to \widehat{W}_n$ is an acyclic Reedy cofibration. However, since A is constant, $L_n A = A$ and therefore $A_n \amalg_{L_n A} L_n\widehat{W} = L_n\widehat{W}$, so it suffices to show is that $L_n\widehat{W} \amalg VW_{n-1} \to \widehat{W}_n$ is an acyclic cofibration. From an argument analogous to the argument in section 4.2, we see that L_n is a term in the coproduct forming $\widehat{W}$, consisting of all of the maps that can factor through W_{n-1}. Using a an analogous argument to that found in ([10], section 2) we see that this map will be an E^2 cofibration.

Now consider the map $\widehat{W} \to B$. We want to show that for all $j \geq 0$ $[\Sigma^j M, \widehat{W}] \to [\Sigma^j M, B]$ is a weak equivalence of simplicial groups. Notice that the higher homotopy groups of $[\Sigma^j M, B]$ are all trivial, as the maps between dimensions are the identity. Analogously to the argument given in ([10], section 2) we see that if there is a map $f : \Sigma^j M \to \widehat{W}_n$ which has $d_i f = 0$ for all $i < n$ there will be a map $g : \Sigma^j M \to \widehat{W}_{n+1}$ such that $d_i f = 0$ for $i \leq n$ and $d_{n+1} g = f$. In particular, this means that $\pi_i[\Sigma^j M, \widehat{W}] = 0$ is trivial, so that the map above will, in fact, be a weak equivalence of simplicial groups.

So now we have constructed a factorization $A \to \widehat{W} \to B$ of any map $A \to B$ between constant objects in $s\mathcal{C}$ into an E^2 cofibration followed by an E^2 weak equivalence. We will use this to show that we can factor any map between $X, Y \in s\mathcal{C}$ into an E^2 cofibration followed by a map which induces an acyclic fibration of simplicial groups

$$[\Sigma^j M, \mathrm{diag}\,\widehat{W}] \to [\Sigma^j M, Y].$$

We know that we can factor $X_n \to Y_n$ into a map $X_n \to \widehat{W}X_n \to Y_n$ with $X \to Z$ an E^2 cofibration and $Z \to Y$ an E^2 weak equivalence. This means that given a map $X \to Y \in s\mathcal{C}_*$ we have a factorization of double simplicial complexes $\widetilde{X} \to \widehat{W}X \to Y$ where the double simplicial complexes X and Y are constant in one direction. We know that for maps in that direction the factorization is into a E^2 cofibration followed by a weak equivalence.

Now we define a complex Z with $Z_n = (\widehat{W}X_n)_n$ (so we restrict to the diagonal in the above double complexes). We claim that the factorization $X \to Z \to Y \in s\mathcal{C}_*$ will have the first map an E^2 cofibration and the second induce an acyclic fibration of simplicial groups $[\Sigma^j M, Z] \to [\Sigma^j M, Y]$. The first map is clearly an E^2 cofibration because the condition for $X_n \to \widehat{W}X_n$ to be an E^2 cofibration will be independent of X, and will be dependent on lower dimensions of Z only by the latching object, which will be dependent only on the diagonal. The second map induces a fibration of simplicial groups by construction, as any map $\Sigma^j M \to Y$ will appear in Z by construction. It will also induce a weak equivalence of groups by the result of [8]. Thus the second map induces an acyclic fibration of simplicial groups, as desired. □

Now we are ready to prove the main theorem of this section.

Theorem 5.11. *The E^2 model category structure given above induces a simplicial model category structure on $s\mathcal{C}_*$.*

Proof. We need to check the model category axioms for this structure.

The first two axioms are easy. MC1 is clear from MC1 in simplicial groups. MC2 is clear from MC2 of $\mathcal{C}$, the Reedy model category structure, and MC2 in simplicial groups.

We will now construct factorizations to satisfy MC4. For the acyclic cofibration-fibration factorization, we use lemma 5.9 to factor the map $X \to Y$ into an acyclic E^2 cofibration followed by a map which induces a fibration on $[\Sigma^j M, X] \to [\Sigma^j M, Y]$. Factor the latter map into a acyclic Reedy cofibration followed by a Reedy fibration. The fibration must induce a fibration on simplicial groups. Thus we have the following diagram:

$$\begin{array}{ccc} & \widetilde{X} & \overset{i}{\underset{\sim}{\hookrightarrow}} & \widetilde{X}' \\ \overset{\sim}{\nearrow} & & \searrow & \downarrow p \\ X & & \longrightarrow & Y \end{array}$$

where i is an acyclic E^2 cofibration because every Reedy acyclic cofibration is an acyclic E^2 cofibration, and where p is an E^2 fibration because it is a Reedy fibration that induces a fibration $[\Sigma^j M, \widetilde{X}'] \to [\Sigma^j M, Y]$. Thus we have a factorization

$$X \overset{\sim}{\hookrightarrow} \widetilde{X}' \twoheadrightarrow Y$$

as desired.

In order to construct the cofibration-acyclic fibration factorization, we do the analogous argument using lemma 5.10 as the initial factorization.

Lastly we consider MC3. Suppose that we have a diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ i \downarrow & & \downarrow{\wr}\, p \\ B & \longrightarrow & Y \end{array}$$

We want to show that there is a lift $B \to X$. First, notice that from the definitions of E^2 cofibration and Reedy fibration, and the fact that $\mathcal{C}$ is a model category, there is a lift in the 0-th dimension. Consider the following diagram:

$$\begin{array}{ccccccccc} \mathrm{sk}_0 A & \longrightarrow \cdots \longrightarrow & \mathrm{sk}_n A & \longrightarrow & \mathrm{sk}_{n+1} A & \longrightarrow \cdots & X \\ \downarrow i_0 & & \downarrow i_n & & \downarrow i_{n+1} & & \downarrow{\wr}\, p \\ \mathrm{sk}_0 B & \longrightarrow \cdots \longrightarrow & \mathrm{sk}_n B & \longrightarrow & \mathrm{sk}_{n+1} B & \longrightarrow \cdots & Y \end{array}$$

If, given a lift $\mathrm{sk}_n B \to X$ we can construct a lift $\mathrm{sk}_{n+1} B \to X$ that is consistent with the previous lift, we will be able to take the colimit of the left-hand side of the diagram and construct a lift $B \to X$ that will make the diagram commute. This is equivalent to getting a lift for the map

$$\mathrm{sk}_{n+1} A \amalg_{\mathrm{sk}_n A} \mathrm{sk}_n B \longrightarrow \mathrm{sk}_{n+1} B.$$

Below the $n+1$-st dimension this is an isomorphism; in the $n+1$-st dimension this is exactly $A_{n+1} \amalg_{L_{n+1} A} L_{n+1} B \to B_{n+1}$. Consider the diagram

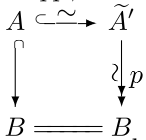
$$\begin{array}{ccc} (A_{n+1} \amalg_{L_{n+1} A} L_{n+1} B) \amalg Z_{n+1} & \longrightarrow & X_{n+1} \\ \wr \downarrow & & \downarrow \\ B_{n+1} & \longrightarrow & M_{n+1} X \times_{M_{n+1} Y} Y_{n+1} \end{array}$$

where the right-hand vertical map is an E^2 fibration because $X \to Y$ is a Reedy fibration. We know that a lift exists in this diagram. In particular, this means that a lift exists in the original diagram, by similar arguments as when we constructed lifts in theorem 4.9. So we have completed the induction, as desired.

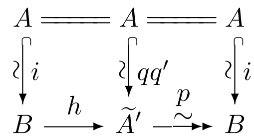
Now consider a diagram with an acyclic E^2 cofibration i and an E^2 fibration p. We factor i into an acyclic E^2 cofibration q followed by an acyclic Reedy cofibration q' and an E^2 fibration p' as we did in the proof of MC4 above. By MC1 p' must be acyclic. Thus we have the following diagram:

$$\begin{array}{ccccccc} A & \overset{q}{\underset{\sim}{\hookrightarrow}} & \widetilde{A} & \overset{q'}{\underset{\sim}{\hookrightarrow}} & \widetilde{A}' & \overset{p'}{\underset{\sim}{\twoheadrightarrow}} & B \\ \downarrow & & & \text{Reedy} & & & \downarrow \\ X & & & \underset{p}{\twoheadrightarrow} & & & Y \end{array}$$

Notice that q has the RLP with respect to all E^2 fibrations by construction, and q' has the RLP with respect to all E^2 fibrations because it is an acyclic Reedy cofibration, and these have the LLP with respect to all Reedy fibrations (and therefore with respect to all E^2 fibrations). Thus it suffices to show that i is a retract of qq', as this will imply that i must also have the LLP with respect to the E^2 fibrations. Notice that in



there is a lift $h : B \to A'$ that makes the diagram commute. But this means that we can construct the following diagram

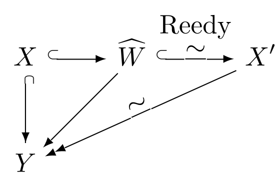


where $ph = \mathrm{id}_B$, so i is a retract of qq', as desired.

Now it only remains to prove that this structure makes $s\mathcal{C}_*$ into a simplicial model category. In order to do this we first prove two lemmas:

Lemma 5.12. *Any E^2 cofibration is a retract of a composition gh, where g is an acyclic Reedy cofibration and h is "strongly M-free", where a map $X \to Y$ is "strongly M-free" if for each $n \geq 0$ there is a Z_n weakly equivalent to a coproduct of $\Sigma^j M$'s with a map $Z_n \to Y_n$ such that $(X_n \amalg_{L_n X} L_n Y) \amalg Z_n \to Y_n$ is an isomorphism.*

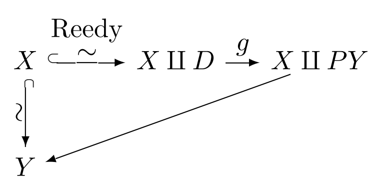
Proof. Consider an E^2 cofibration $X \to Y$. We can factor this into an E^2 cofibration followed by a map that induces an acyclic fibration on simplicial groups $[\Sigma^j M, X] \to [\Sigma^j M, Y]$, as in proposition 5.10. Factor this second map into an acyclic Reedy cofibration followed by a Reedy fibration. By MC1 the Reedy fibration must be an acyclic E^2 fibration. Thus we have a diagram



However, by MC3, this means that we have a lift $Y \to X'$; in particular, this means that $X \to Y$ is a retract of $X \to X'$. Note that by construction the map $X \to \widehat{W}$ is strongly M-free. So we are done. ☐

Lemma 5.13. *Any acyclic E^2 cofibration is a retract of a composition fgh, where f and h are acyclic Reedy cofibrations and g is a homotopy equivalence.*

Proof. We can factor any map $X \to Y$ into an acyclic E^2 cofibration $X \to X \amalg PY$ followed by a map $X \amalg PY \to Y$ that induces a fibration of simplicial groups as in 5.9. Notice that the map $X \to X \amalg PY$ can be factored into a map $X \to X \amalg D \to X \amalg PY$ where D is the coproduct over all pairs (a, b, n, j) (as in the definition of PY) of $C\Sigma^j M \otimes \Delta^0$. Then we have a diagram



Notice that if we factor the map $X \amalg PY \to Y$ into an acyclic Reedy cofibration $X \amalg PY \to Y'$ followed by a Reedy fibration (which will be an E^2 fibration) $Y' \to Y$ there will be a lift $Y \to Y'$ that means that $X \to Y$ is a retract of $X \to Y'$. But $X \to Y'$ is a composition of exactly the right form (as $X \amalg D \to X \amalg PY$ is a a simplicial homotopy equivalence since it is the cobase change of a simplicial homotopy equivalence), so we are done. ☐

Let $i : K \to L \in s\mathbf{Sets}$ and $f : X \to Y \in s\mathcal{C}$. We will denote the induced map

$$X \otimes L \amalg_{X \otimes K} Y \otimes K \longrightarrow Y \otimes L$$

by $f \otimes i$.

We are going to show that SM7b holds. If $f \otimes i$ is an E^2 cofibration then every diagram

$$\begin{array}{ccc} X \otimes L \amalg_{X \otimes K} Y \otimes K & \longrightarrow & E \\ {\scriptstyle f \otimes i}\downarrow & & \wr\downarrow{\scriptstyle p} \\ Y \otimes L & \longrightarrow & B \end{array}$$

has a lift. By adjointness, this means that f must have the LLP with respect to all maps

$$\begin{array}{ccc} X & \longrightarrow & E^L \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p^i} \\ Y & \longrightarrow & X^K \times_{Y^K} Y^L \end{array}$$

(with the obvious modification if we want $f \otimes i$ to be acyclic). Thus in particular notice that if we can show that a composition of maps satisfies these lifting properties, then the map itself does.

First, suppose that f is an E^2 cofibration. We know that we can write f as a composition gh with g an acyclic Reedy cofibration and h a strongly M-free map. Notice that since p is a Reedy fibration, from proposition 4.13 we know that p^i is a Reedy fibration, and that therefore g, the acyclic Reedy cofibration, has the left lifting property with respect to it. Thus it remains to show that this holds for a strongly M-free map. We know that the map

$$(X_n \amalg_{L_n X} L_n Y) \amalg Z_n \longrightarrow Y_n$$

is an isomorphism. Consider the following diagram, analogous to the diagram we considered in the proof of MC3:

$$\begin{array}{ccccccccc} \mathrm{sk}_0 X & \longrightarrow \cdots \longrightarrow & \mathrm{sk}_n X & \longrightarrow & \mathrm{sk}_{n+1} X & \longrightarrow & \cdots & & E \\ \downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle h_n} & & \downarrow{\scriptstyle h_{n+1}} & & & & {\scriptstyle p^i}\downarrow \\ \mathrm{sk}_0 Y & \longrightarrow \cdots \longrightarrow & \mathrm{sk}_n Y & \longrightarrow & \mathrm{sk}_{n+1} Y & \longrightarrow & \cdots & & B \end{array}$$

By analogous reasoning we can reduce this problem to the problem of constructing a lift for the map

$$\mathrm{sk}_{n+1} X \amalg_{\mathrm{sk}_n X} \mathrm{sk}_n Y \longrightarrow \mathrm{sk}_{n+1} Y.$$

Below the $n+1$-st dimension these maps are isomorphisms, so the lift exists by assumption. At the $n+1$-st dimension this is exactly the map

$$X_{n+1} \amalg_{L_{n+1} X} L_{n+1} Y \longrightarrow Y_{n+1}.$$

Since $Y_{n+1} \cong (X_{n+1} \amalg_{L_{n+1} X} L_{n+1} Y) \amalg Z_{n+1}$, this map is the cobase change of the map $* \to Z_{n+1} \otimes \Delta^{n+1}/\partial\Delta^{n+1}$. Thus it suffices to show that

$$(Z_{n+1} \otimes \Delta^{n+1}/\partial\Delta^{n+1}) \otimes K \to (Z_{n+1} \otimes \Delta^{n+1}/\partial\Delta^{n+1}) \otimes L$$

is an E^2 cofibration, which is acyclic if i is. Note that this map is clearly an E^2 cofibration, since it is a coproduct of Z_{n+1}'s going to a coproduct of Z_{n+1}'s. If i is one of the maps $\Delta^0 \to \Delta^1$ (the only acyclic cofibrations we need to check to show SM7b) it is an E^2 weak equivalence by corollary 5.8. Thus $f \otimes i$ is an (acyclic) E^2 cofibration, as desired.

Now it remains to show that $f \otimes i$ is an E^2 weak equivalence if f is an acyclic E^2 cofibration. We know that we can factor f into $f_0 gh$ where f_0 and h are acyclic Reedy cofibrations, and g is a simplicial homotopy equivalence. Thus it suffices to show that a lift exists in the above diagram for g, which clearly works since it is a simplicial homotopy equivalence. So we are done. □

Notice that the above theorem shows that it is possible to slightly modify the Reedy model category structure, in particular by making the class of weak equivalences larger, that makes $s\mathcal{C}_*$ into a simplicial model category. However, this structure has many more limitations than the Reedy model category structure: we needed $\mathcal{C}$ to be pointed, to have arbitrary limits and colimits, and for all objects to be fibrant.

However, these conditions are not as strong as they seem at first glance. They work for simplicial topological spaces, for example, and they also work for the example that we have been looking at: simplicial objects over $\mathbf{Ch}_R$.

5.2 Example: $s\mathbf{Ch}_R$ and Double Chain Complexes

Notice that in $\mathbf{Ch}_R$ everything is fibrant (since fibrations are surjections, and any map to the trivial chain complex is a surjection). $\mathbf{Ch}_R$ is pointed, as the chain complex with only trivial modules has exactly one map into and from every other chain complex. In addition, as the category of modules over R has arbitrary limits and colimits, $\mathbf{Ch}_R$ will also have arbitrary limits and colimits. Thus we see that there exists an E^2 model category structure on $s\mathbf{Ch}_R$, and thus also on the category of double chain complexes.

First, we need to choose a cofibrant cogrouplike object M that will be the basis for the E^2 model category structure. Notice that, in $s\mathbf{Top}_*$, if $M = S^1$ then $\Sigma^j M = S^{j+1}$ and $[\Sigma^j M, X] = \pi_j(X)$; in particular, the weak equivalences are the ones that are weak equivalences on the simplicial groups $\pi_j(X)$ for each j. With this motivation we want to choose an M that is analogous to S^1: a cycle in the 1-st dimension, where each suspension will simply make the cycle a dimension higher.

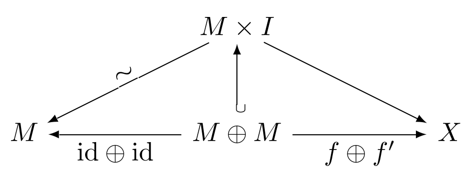
Proposition 5.14. *The chain complex M with R in the 1-st dimension and 0 in all others, along with the homotopy class of the diagonal map $\tau : M \to M \oplus M$ is a cofibrant cogrouplike object.*

In this section we will only construct one type of homotopy between maps $M \to X$. Since M is cofibrant and X is fibrant, this will be enough to show that two maps are homotopic.

Before proving the proposition, we prove a lemma.

Lemma 5.15. *Let $f, f' : M \to X$ be two maps, and let σ, σ' be the cycles they pick out in X_1. Then $f \sim f'$ if and only if $[\sigma] = [\sigma']$ in $H_1(X)$.*

Proof. Suppose that $f \sim f'$; that means that we have a cylinder object $M \times I$ such that the following diagram commutes:



First, consider the left triangle. The vertical map from $M \oplus M$ picks out two cycles in $M \times I$. Since the composition with the weak equivalence on the left is an isomorphism in homology, and the two cycles map to the same cycle in M, they must have been homologous in $M \times I$. In particular, this means that σ and σ' must be homologous in X.

Now suppose that $[\sigma] = [\sigma']$. This means that there is a $\tau \in X_2$ such that $d\tau = \sigma - \sigma'$. Let $M \times I$ be a chain complex with $R \oplus R$ in the first dimension, and R in the second; the image of the second dimension is the difference of the two generators in the first; this clearly defines a cylinder object, where the map into the cylinder object maps the two cycles in $M \oplus M$ into different generators. Map the generator of the second dimension of M into τ. This construction produces a left homotopy equivalence between f and f'. ☐

Proof of proposition. First, note that M is clearly cofibrant, as it is a free module over R. Thus it suffices to show that τ induces a group structure on $[M, Y]$ for all $Y \in \mathbf{Ch}_R$. This group structure is defined by

$$[f] * [g] = [(f \oplus g)\tau].$$

Notice that the definition of the operation on $[M, Y]$ is simply addition of cycles in H_1. In particular, from lemma 5.15 we see that replacing maps by homotopic maps is simply equivalent to replacing cycles by homologous cycles, which we know makes addition of cycles well-defined. Then the group axioms are all clear from the fact that homology is a group.

□

As both fibrations and weak equivalences rely on $[\Sigma^j M, X]$, our next step in computing the model category structure is to compute the suspensions of M, as well as the simplicial groups $[\Sigma^j M, X]$. It is a simple computation to show that $\Sigma^j M$ has R in the $j+1$-st dimension and 0 everywhere else. Notice that in light of proposition 5.14 we know that each $\Sigma^j M$ is a cogrouplike object. In addition, notice that a proof analogous to the proof of proposition 5.15, proves the following corollary:

Corollary 5.16. *For all $j, n \geq 0$, $[\Sigma^j M, X_n] = H_{j+1}(X_n)$.*

In order for a map $X \to Y \in s\mathbf{Ch}_R$ to be a weak equivalence, we need the induced map $[\Sigma^j M, X] \to [\Sigma^j M, Y]$ to be a weak equivalence of simplicial groups for all $j \geq 0$. From our computations above, we see that in fact this means that $H_j(X) \to H_j(Y)$ should be a weak equivalence of simplicial groups for all $j \geq 1$.

Now consider the E^2 fibrations. Every E^2 fibration is a Reedy fibration, meaning that it is a surjective map that is an isomorphism in horizontal homology above the first row. In addition, we must have, for each $j \geq 0$, a fibration of simplicial groups $[\Sigma^j M, X] \to [\Sigma^j M, Y]$. Then for each $j \geq 0$ we have $H_j(X) \to H_j(Y)$ a surjection. However, since the Reedy condition already implies this, we see that the fibrations are the maps that are surjective and isomorphisms in horizontal homotopy above the first row, and are surjective in homotopy on the first row.

After normalization, simplicial groups in the rows become chain complexes. In particular, the condition that after taking vertical homology the map is a weak equivalence of simplicial groups becomes the condition that a weak equivalence must be an isomorphism in "homology of double chain complexes", where we take vertical homology first and then horizontal homology. The E^2 fibrations become simply maps that are surjections and isomorphisms in horizontal homology above the first row, and surjections in the first row.

Thus we see that the E^2 model category structure on double chain complexes is that a map $f : X \to Y \in \mathbf{dCh}_R$ is

- a weak equivalence if it induces an isomorphism on $H_i^{\leftarrow} H_j^{\downarrow}(X) \to H_i^{\leftarrow} H_j^{\downarrow}(Y)$ for all $i, j \geq 0$,
- a fibration if it is a surjection and an isomorphism in horizontal homology above the first row and a surjection in horizontal homology in the first row, and
- a cofibration if it has the LLP with respect to the acyclic fibrations.

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