Hilbert's Third Problem

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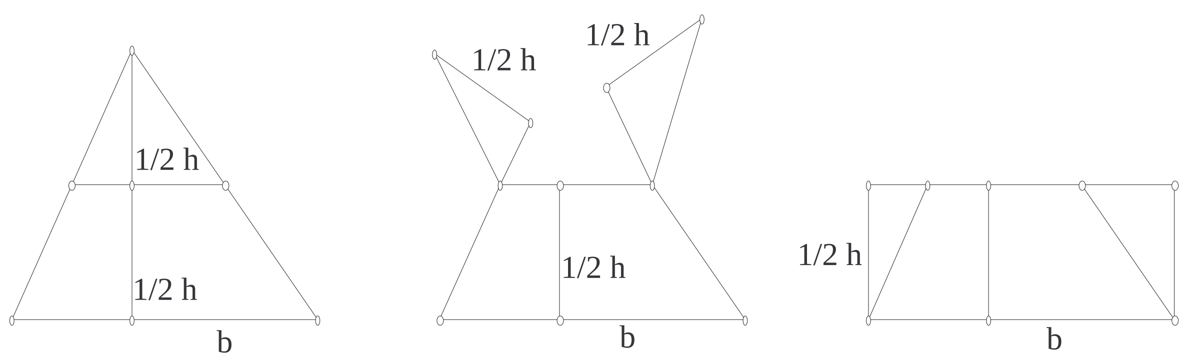
1 Polygons

Definition 1 *Two polygons with equal area are* **CAC** *(Congruent After Cutting) if they can be cut up into a finite number of congruent parts. Equivalently, two polygons are CAC if one can be cut up into a finite number of polygons and rearranged into the other. The notation* $F \simeq G$ *will be used when two polygons* F *and* G *are CAC.*

For example, the following illustrates that a triangle can be CAC with a rectangle:



1.1 Transitivity

Theorem 1 *For three polygons* A, B, C *if* $A \simeq B$ *and* $B \simeq C$ *then* $A \simeq C$.

To show this, take B and cut it up into pieces that can be rearranged to form C. Over those cuts, make the ones that can be rearranged to form A. Then, out of the resulting pieces both A and C can be formed. This shows that $A \simeq C$.

1.2 Triangles

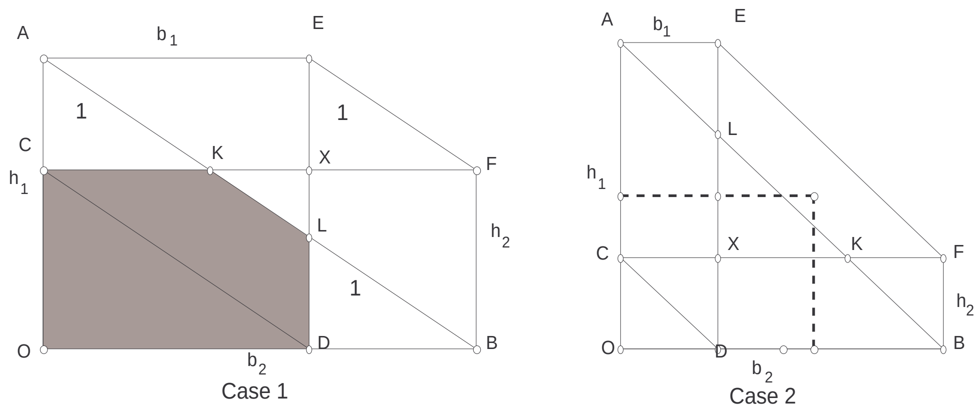
Theorem 2 *Any triangle is CAC to some rectangle.*

The picture above can be duplicated for any triangle. From the vertex with the largest angle draw the altitude to the other side, and the medial line parallel to that side. Cutting along these lines, and rearranging the pieces as in the above picture yeilds that any triangle is COC to some rectangle.

1.3 Rectangles

Theorem 3 *Any two rectangles of equal area are CAC.*

Consider any two rectangles $OAED$ and $OCFB$ with equal area. Then, if the dimensions of $OAED$ are b_1, h_1 and the dimensions of $OCFB$ are b_2, h_2, $b_1h_2 = b_2h_2$.



Since $b_1h_1 = b_2h_2$, $\frac{h_1}{b_2} = \frac{h_2}{b_1}$, so triangles OAB and OCD are similar. Thus $AB \| CD$. Also, because rectangle $OCXD$ is part of both rectangles, the areas of rectangles $CAEX$ adn $DXFB$ are equal, so $(h_1 - h_2)b_1 = (b_2 - b_1)h_2$. Thus $\frac{h_1 - h_2}{h_2} = \frac{b_2 - b_1}{b_1}$, so triangles OCD and XEF are similar. So $AB \| EF$.

Since $AEFK$ is a parallelogram, $AE = KF$. Also, since $AEXC$ is a rectangle $AE = CX$. Thus $KF = CX$. Since their sides are parallel, and $AC = EX$ $\triangle ACK \cong \triangle EXF$. Analogously, $\triangle LDB \cong \triangle EXF$. Also, since $AE = KF$, $EL = FB$, and $AK = LB$, $\triangle AEL \cong \triangle KFB$.

If line AB intersects rectangle $OCXD$ that means that $AB \leq CX + KF$, so $2b_1 \geq b_2$. In this case it is clear that rectangles $OAED$ and $OCFB$ are CAC, since they both consist of the shaded pentagon, one of the congruent regions marked "1", and one of triangles AEL or KFB.

If line AB doesn't intersect rectangle $OCXD$ then a different construction is necessary. Mark off segment OD on OB as many times as is necessary so that the midpoint of OB is inside the marked off area. Then break up segment AO into that many segments, and cut rectangle $OAED$ onto smaller rectangles at those marks. Rearrange those along the marked off segments on OB. The result is a rectangle, in which $2b_1 \geq b_2$, and that rectangle can be cut up into equal parts with rectangle $OCFB$. By transitivity of CAC, $OAED \simeq OCFB$.

Thus any two rectangles are CAC.

1.4 Polygons

Theorem 4 *Any two polygons of equal area are CAC.*

Any polygon can be cut into triangles, and each of those triangles is CAC to some rectangle. For an arbitrary length x each rectangle is CAC to some rectangle with one side equal to x. Thus the polygon is CAC to some rectangle with one side equal to x.

If two polygons have equal area A, then each will be CAC to a rectangle with one side x and the other $\frac{A}{x}$. Thus they are CAC to one another.

2 Polyhedra

Definition 2 *Two polyhedra with equal volume are* **CAC** *if they can be cut up into a finite number of congruent polyhedra.*

After proving that any two polygons of equal area are CAC, it is natural to ask the same question about polyhedra. So,

> If P and Q are two polyhedra of equal volume, are they CAC?

This is Hilbert's Third Problem, and the first to be solved.

2.1 Note on Congruency

When discussing polygons, it is customary to consider two polygons with congruent if they can be placed on one another by a series of rotations, translations, and reflections. However, this becomes more complicated when dealing with polyhedra, for there are two ways to define congruency.

Definition 3 *The group D is the group of all rigid motions of $\mathbb{R}^3$. The group D_0 is the group of all rigid motions of $\mathbb{R}^3$ that preserve orientation.*

Definition 4 *Two polyhedra are* G**-congruent** *if they can be mapped to one another with the motions in group G. Two polyhedra are* G**-CAC** *if they can be cut up into a finite number of polyhedra which are G-congruent.*

Theorem 5 *If two polyhedra are D-congruent then they are D_0-CAC.*

Since the polyhedrons are D-congruent they can be cut into pieces that are D-congruent. Suppose that one of these is A (with it's D-congruent pair A') and it has a plane of symmetry. Let $f \in D$ be the motion that maps A to A', and let s be the reflection of A in it's plane of symmetry. Then one of f and $f \circ s$ preserves orientation, and is thus is in D_0. However, since s maps A to A, $f \circ s$ also maps A to A'. So A and A' are D_0-congruent.

Thus if it can be shown that any polyhedron can be cut into polyhedra that have a plane of symmetry then the theorem will be proven.

In order to do this, first cut the polyhedron along the planes that contain its faces, to obtain some convex polyhedrons. Each of these can be cut into pyramids by selecting any point inside the polyhedron and cutting it into pyramids that have that point as the vertex and a face for the base. Then, each of those pyramids can be cut up into tetrahedrons by triangulating hte base and making pyramids with each triangle as the base and the vertex of the pyramid as a vertex. So it suffices to show that any tetrahedron can be cut into polyhedra that have a plane of symmetry.

Take a tetrahedron $ABCD$, and the sphere inscribed in it. Let A', B', C', D' be the tangency points of the sphere to the faces of the tetrahedron (with A' opposite A, etc.), and let O be the center of the sphere. Cut the tetrahedron into convex polyhedrons $OA'B'CD$, $OA'BC'D$, $OA'BCD'$, $OAB'C'D$, $OAB'CD'$, $OABC'D'$. Consider polyhedron $OA'B'CD$. Since O is equidistant from the planes containing ACD and BCD it is in the plane containing edge CD and bisecting the dihedral angle at that edge. Thus A' and B', the projections of O onto ACD and BCD are also symmetric across that plane, and so that is a plane of symmetry of that polyhedron.

This poves the theorem.

A corrolary of the theorem is that it does not matter for the discussion of CAC whether D or D_0 is used. From here on, D_0-CAC will be referred to as just CAC.

2.2 Notation and Definitions

Definition 5 *A set of numbers $M = \{x_1, x_2, \ldots, x_k\}$ will be called* **dependent** *if there exist $n_1, n_2, \ldots, n_k \in \mathbb{Z}$ not all zero, such that*

$$n_1 x_1 + n_2 x_2 + \cdots + n_k x_k = 0.$$

Definition 6 *Let S be a set of real numbers. A function $f : S \to \mathbb{R}$ is* **additive** *if, for any dependent subset M of S*

$$n_1 f(x_1) + n_2 f(x_2) + \cdots + n_k f(x_k) = 0.$$

In polyhedron A, the set $\{a_1, a_2 \ldots, a_r\}$ will denote the set of lengths of edges, and the set $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ will denote the set of dihedral angles of the polyhedron, such that α_i is the measure of the angle at edge a_i.

Definition 7 *If f is some additive function, then*

$$f(A) = a_1 f(\alpha_1) + a_2 f(\alpha_2) + \cdots + a_r f(\alpha_r).$$

2.3 Additive Functions on Polyhedra

Theorem 6 *Suppose that an additive funtion f such that $f(\pi) = 0$ is defined on the set of dihedral angles of polyhedra $P_1, P_2, \ldots, P_n$ and A, and suppose*

$$A = P_1 + P_2 + \cdots + P_n.$$

Then

$$f(A) = f(P_1) + f(P_2) + \cdots + f(P_n).$$

Cut A up into $P_1, P_2, \ldots, P_n$. Call any place where two edges meet, or where there is a vertex of A or some P_i a vertex. Call any edge of one of the polyhedra connecting two vertices a "link". If l is the length of a link, and α is sun of the dihedral angles of the polyhedrons at that link, then let $lf(\alpha)$ be the weight of that link.

Suppose that n polyhedra meet at a link with length l and with dihedral angles $\alpha_1, \alpha_2, \ldots, \alpha_n$. Then the weight of that link is $lf(\alpha_1+\alpha_2+\cdots+\alpha_n) = lf(\alpha_1)+lf(\alpha_2)+\cdots+lf(\alpha_n)$, which is the sum of the weights of that link with respect to each of the polyhedra. If an edge of A with dihedral angle α is made up of n links with lengths $l_1, l_2, \ldots, l_n$ then the weight of that edge would be $(l_1+l_2+\cdots+l_n)f(\alpha) = l_l f(\alpha)+l_2 f(\alpha)+\cdots+l_n f(\alpha)$, which is also the sum of the weights of the links.

Thus the sum of the weights of the links lying along the edges of A is clearly $f(A)$, so all that is left to prove is that the sum of the weights of all of the other links is 0. There are two other places that there can be links: on the faces of A, and on the interior of A. If a link is on the face of A then the sum of the dihedral angles around it is π, so it's weight is 0. If a link is on the interior of A then the sum of the angles of the polyhedrons around it is 2π, so the weight of that link is 0. Thus the sum of the weights of all of the links is $f(A)$.

2.4 Dehn's Theorem

Theorem 7 *For two polyhedra A and B, if there exists an additive function f on the dihedral angles of the polyhedra such that $f(\pi) = 0$ and $f(A) \neq f(B)$ then A and B are not CAC.*

Suppose that there exists such a function but A and B are CAC. Cut them up into the polyhedrons $P_1, P_2, \ldots, P_n$ that can be rearranged into them both. Then

$$f(A) = f(P_1) + f(P_2) + \cdots + f(P_n) = f(B).$$

This is a contradiction. Thus A and B are not CAC.

2.5 Examples

Example 1 *A cube and a regular tetrahedron are not CAC.*

The proof of this requires the proof of another theorem first.

Theorem 8 *For any natural number $n \geq 3$ the number $\frac{1}{\pi}arccos\frac{1}{n}$ is irrational.*

Let $\varphi = \arccos\frac{1}{n}$. Suppose that $\varphi/\pi = l/k$, with k, l integers. Then $k\varphi = l\pi$, which means that $\cos k\varphi = \pm 1$, a.k.a. that it is an integer. By the addition law for cosines,

$$\cos(k+1)\varphi + \cos(k-1)\varphi = 2\cos k\varphi \cos\varphi,$$

and since $\cos\varphi = \frac{1}{n}$

$$\cos(k+1)\varphi = \frac{2}{n}\cos k\varphi - \cos(k-1)\varphi.$$

Case 1: n is odd. Then $\cos k\varphi = \frac{a}{n^k}$ with $(a,n)=1$. This will be proven with induction on k. When $k=1$ $\cos\varphi = \frac{1}{n}$ and when $k=2$ $\cos 2\varphi = 2\cos^2\varphi - 1 = \frac{2-n^2}{n^2}$. Then

$$\cos(k+1)\varphi = \frac{2}{n}\frac{a}{n^k} - \frac{b}{n^{k-1}} = \frac{2a-bn^2}{n^{k+1}}$$

and since $(a,n)=1$ and $(2,n)=1$ $(2a-bn^2,n)=1$, proving the hypothesis.

Case 2: n is even. Then $n=2m$ where $m \geq 2$. $\cos k\varphi = \frac{a}{2m^k}$ with $(a,2m)=1$. This can be proven with an inductive step similar to the induction above.

This shows that $\cos k\varphi$ can never be an integer, and so $\frac{\varphi}{\pi}$ can nver be rational.

Now let $\alpha = \arccos\frac{1}{3}$. This is the dihedral angle between two of the sides of the regular tetrahedron. Then the set on which the function f needs to be defined is $\{\pi, \frac{\pi}{2}, \alpha\}$. Define f by

$$f(\pi)=0, \quad f(\frac{\pi}{2})=0, \quad f(\alpha)=1.$$

If there is some dependency $n_1\pi + n_2\cdot\frac{\pi}{2} + n_3\alpha = 0$ with $n_3 \neq 0$ we get that

$$\frac{1}{\pi}\arccos\frac{1}{3} = \frac{\alpha}{\pi} = -\frac{n_1+\frac{n_2}{2}}{n_3}$$

which contradicts theorem 8. So $n_3=0$, and so clearly $n_1f(\pi)+n_2f(\frac{\pi}{2})+n_3f(\alpha)=0$, which means that f is additive. Let the length of the edges of the cube is l and the length of the edges of the tetrahedron be m. Then

$$f(\text{cube}) = 12lf(\frac{\pi}{2}) = 0.$$

and

$$f(\text{tetrahedron}) = 6mf(\alpha) = 6m \neq 0$$

so the cube and the tetrahedron are not CAC.

Example 2 *A cube and a right tetrahedron are not CAC.*

Let α be the angle between the largest face of the right tetrahedron and another face. Since $\cos\alpha = \frac{1}{\sqrt{3}}$, it can be shown (with an induction similar to that in theorem 8) that $\frac{\alpha}{\pi} = \frac{1}{\pi}\arccos\frac{1}{\sqrt{3}}$ is always irrational. If f is once again defined by

$$f(\pi)=0, \quad f(\frac{\pi}{2})=0, \quad f(\alpha)=1$$

then $f(\text{cube})=0$. If the length of the shorter edges of the right tetrahedron is l, then the length of the other three is $l\sqrt{2}$, and $f(\text{tetrahedron}) = 3lf(\frac{\pi}{2}) + 3l\sqrt{2}f(\alpha) = 3l\sqrt{2} \neq 0$.