A Generalization of Wigner's Law

Inna Zakharevich*

June 12, 2005

Abstract

We present a generalization of Wigner's semicircle law: we consider a sequence of probability distributions $(p_1, p_2, \dots)$, with mean value zero and take an $N \times N$ real symmetric matrix with entries independently chosen from p_N and consider analyze the distribution of eigenvalues. If we normalize this distribution by its dispersion we show that as $N \to \infty$ for certain p_N the distribution weakly converges to a universal distribution. The result is a formula for the moments of the universal distribution in terms of the rate of growth of the k-th moment of p_N (as a function of N), and describe what this means in terms of the support of the distribution. As a corollary, when p_N does not depend on N we obtain Wigner's law: if all moments of a distribution are finite, the distribution of eigenvalues is a semicircle.

1 Introduction

In this paper we study the density of states of random real symmetric matrices of very large dimension with i.i.d. entries from a mean-0 distribution. Such problems arise in nuclear physics, for example in the descriptions of the interactions of heavy nuclei (see [7], [2] for sources). Using the method of moments, Wigner showed that the expected distribution of eigenvalues of such a matrix is a semicircle, provided that all moments of the probability distribution from which the entries are selected exist (see [13], [14], [5]). Other methods were used to show that, for $\delta > 0$, the presence a finite $(2 + \delta)$-th

*Email:zakharev@fas.harvard.edu

moment is all that is necessary for convergence to the semicircle, and to generalize this to matrices where the entries were not necessarily independently distributed (see [7]). In [11] and [12] Tracy and Widom showed that (after rescaling) the largest eigenvalues of matrices in the GOE, GUE, and GSE all converge to the same universal distribution; Soshnikov later extended this to all real symmetric matrices with i.i.d. entries from symmetric distributions with rapidly decaying tails (see [8]).

However, all of these papers assume reasonably nice distributions; they all have finite second moments. These theories do not apply to some distributions of entries with heavy tails, such as the Cauchy distribution. We are interested in studying the density of states of matrices with entries chosen from such distributions. In [10], Soshnikov and Fyodorov showed that the largest eigenvalue of A^tA, where A is an $m \times n$ matrix with entries independently chosen from the Cauchy, does not follow the Tracy-Widom law, but instead follows a Poisson law; Soshnikov later showed that this applies to random symmetric matrices with entries chosen from distributions with heavy tails which drop off approximately as $1/x^\alpha$ (see [9]).

We wish to study the density of eigenvalues in random real symmetric matrices with entries i.i.d. from a distribution with heavy tails. In this paper we apply Wigner's original method, the method of moments, to general mean-0 probability distributions. We do this by computing moments of the eigenvalue distribution of an $N \times N$ real symmetric matrix, and then taking the limit as $N \to \infty$. We take a sequence of distributions $(p_1, p_2, \ldots)$, which may be thought of as a sequence of approximations to a target distribution p, and select the entries of the $N \times N$ matrix from p_N. We assume that each p_i has mean 0 and all higher moments finite; however, the sequence can converge to a distribution with infinite higher moments. For example, p_N could be a truncation of a distribution with infinite moments.

Definition 1. Let $\mu_N(k)$ be the k-th moment of p_N. Let A_N be an $N \times N$ real symmetric matrix with entries chosen independently from p_N. Let

$$C_k = \lim_{N\to\infty} \frac{\mu_N(k)}{N^{k/2-1}\mu_N(2)^{k/2}}$$

if the limit exists.

If C_k exists and is finite for all k, then it is possible to calculate a formula for the moments of the distribution of the eigenvalues of A_N in the limit as $N \to \infty$. We shall prove

Theorem 1. *Suppose the C_k exist and are finite. Then there exists a universal distribution that the distribution of eigenvalues of these matrices weakly converges to as $N \to \infty$. This distribution satisfies the following properties:*

1. *the distribution is symmetric;*
2. *if $C_k = O(\alpha^k)$ for some constant α, then the distribution is uniquely determined by its moments;*
3. *in the special case that $C_k = 0$ for all $k > 2$, the distribution is the semicircle;*
4. *in all other cases the distribution has unbounded support.*

Part 3 implies Wigner's semicircle law, because if all moments are finite, then $C_k = 0$ for $k > 2$.

We then apply this theory to two examples: the case where p_i is a truncation of a distribution of the form $C/(1 + |x|^m)$ and the case of adjacency matrices of approximately k_N-regular graphs where nonzero entries are allowed to be ± 1. In the former case we find that if $1 < m < 3$ there is a truncation that results in a distribution that is not the semicircle; if $m \geq 3$, that is impossible. In the latter case we find that if $k_N \to \infty$ as $N \to \infty$ we obtain a semicircle; in other cases we obtain a distribution that is not the semicircle.

2 A Calculation of Moments of an Eigenvalue Distribution

Let A_N be a real symmetric matrix with eigenvalues $\lambda_1(A_N), \ldots, \lambda_N(A_N)$. To each matrix we can associate a probability distribution μ_{A_N} by

$$\mu_{A_N}(x) = \frac{1}{N} \sum_{i=1}^{N} \delta \left(x - \frac{\lambda_i(A_N)}{\sqrt{N \mu_N(2)}} \right)$$

where δ is the Dirac delta function. Let $\mathbb{E}_{p_N}[x]$ denote the expected value of x with respect to the probability distribution p_N. Let $\mathbb{E}[x]_{A_N}$ denote the expected value of x with respect to μ_{A_N}. Note that if we take the expected value of x with respect to the probability distribution μ_{A_N} (the mean of μ_{A_N})

and then average over all matrices A_N, we will have the mean of the expected distribution of the eigenvalues. Similarly, if we take the kth moment of x over μ_{A_N} and then average over all matrices A_N, we will get the kth moment of the expected distribution of eigenvalues. We define

$$P(A_N)dA_N = \prod_{i\leq j} p_N(a_{ij})da_{ij},$$

the probability distribution for the A_N. We will denote the expected value of x with respect to the probability function $P(A_N)$ by $\mathbb{E}[x]$.

First note the following:

$$\begin{aligned}
\mathbb{E}[x^k]_{A_N} &= \int_{-\infty}^{\infty} x^k \mu_{A_N}(x)dx \;=\; \frac{1}{N}\sum_{i=1}^{N}\left(\frac{\lambda_i(A_N)}{\sqrt{N\mu_N(2)}}\right)^k \\
&= \frac{1}{N\sqrt{N\mu_N(2)}^k}\text{Trace}(A_N^k) \\
&= \frac{1}{N\sqrt{N\mu_N(2)}^k}\sum_{i_1=1}^{N}\cdots\sum_{i_k=1}^{N} a_{i_1i_2}a_{i_2i_3}\cdots a_{i_ki_1}. \qquad (1)
\end{aligned}$$

Then

$$\begin{aligned}
\mathbb{E}[\mathbb{E}[x^k]_{A_N}] &= \int_{-\infty}^{\infty}\mathbb{E}[x^k]_{A_N}P(A_N)dA_N \\
&= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{1}{N\sqrt{N\mu_N(2)}^k}\sum_{i_1,\dots,i_k=1}^{N} a_{i_1i_2}\cdots a_{i_ki_1}\prod_{i\leq j}p_N(a_{ij})da_{ij} \\
&= \frac{1}{N\sqrt{N\mu_N(2)}^k}\sum_{i_1,\dots,i_k=1}^{N}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} a_{i_1i_2}\cdots a_{i_ki_1}\prod_{i\leq j}p_N(a_{ij})da_{ij} \\
&= \sum_{i_1,\dots,i_k=1}^{N}\frac{1}{N\sqrt{N\mu_N(2)}^k}\mathbb{E}_{p_N}[a_{i_1i_2}\cdots a_{i_ki_1}]. \qquad (2)
\end{aligned}$$

3 Calculation of Moments by Magnitude Analysis

All order computations will be done as $N \to \infty$. In addition, since we only care about the moments in this limit, we will always assume that $N > k$.

Definition 2. Fix an $i_1, \ldots, i_k$ in (1); this will fix a term in that sum. Let the matrix $S_{i_1,\ldots,i_k}$ have its ij-th entry $(i \leq j)$ equal to the number of times that a_{ij} or a_{ji} appears in $a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}$, and all other entries 0. Call this matrix the *associated matrix* of the term. Define

$$E(S_{i_1,\ldots,i_k}) = \mathbb{E}_{p_N}\left[\prod_{i\leq j} a_{ij}^{s_{ij}}\right].$$

Two associated matrices will be called *similar* if the entries of one are a permutation of the entries of the other.

Definition 3. Let $T_{N,k}$ be the set of matrices associated with terms in (1). By construction, the map from terms in (1) to elements of $T_{N,k}$ is injective. Notice that if $S_{i_1,\ldots,i_k}$ is similar to $S_{j_1,\ldots,j_k}$ then $E(S_{i_1,\ldots,i_k}) = E(S_{j_1,\ldots,j_k})$. Let $\mathcal{S}_{N,k}$ be the set of equivalence classes of $T_{N,k}$ (where two matrices are equivalent if they are similar). Then we can define $E([S_{i_1,\ldots,i_k}]) = E(S_{i_1,\ldots,i_k})$ for any equivalence class $[S_{i_1,\ldots,i_k}]$.

For conciseness, we will sometimes refer to a term with associated matrix in an equivalence class $\sigma \in \mathcal{S}_{N,k}$ as a term in σ.

Note that we can rewrite (2) as

$$\sum_{\sigma\in\mathcal{S}_{N,k}} \frac{1}{N^{k/2+1}\mu_N(2)^{k/2}} \#\{S \in T_{N,k} : [S] = \sigma\} E(\sigma). \tag{3}$$

Notice that the set $\mathcal{S}_{N,k}$ is independent of N, since an equivalence class of matrices is uniquely defined by its nonzero entries, and those are only dependent on k. Thus the number of terms in the above sum is independent of N, and we can calculate the limit of this sum by calculating the limit of each term as $N \to \infty$ and then summing the limits.

Consider a $\sigma \in \mathcal{S}_{N,k}$ such that the matrices in that equivalence class have b nonzero entries, $s_1, \ldots, s_b$. Note that because of the way that the indices of the a_{ij} are arranged in the product, there can be at most $b+1$ distinct indices in each term in σ. Thus we see that there will be $L_\sigma N^{b+1} + O(N^b)$ (for some constant L_σ) matrices with terms in σ. Note that

$$\begin{aligned}\frac{\#\{S \in T : [S] = \sigma\}E(\sigma)}{N^{k/2+1}\mu_N(2)^{k/2}} &= \frac{(L_\sigma N^{b+1} + O(N^b))E(\sigma)}{N^{k/2+1}\mu_N(2)^{k/2}} \\ &= \frac{(L_\sigma N^{b+1} + O(N^b))}{N^{b+1}} \frac{\mu_N(s_1)}{N^{s_1/2-1}\mu_N(2)^{s_1/2}} \cdots \frac{\mu_N(s_b)}{N^{s_b/2-1}\mu_N(2)^{s_b/2}}\end{aligned}$$

since $E(S_N) = \mu_N(s_1) \cdots \mu_N(s_b)$ and $s_1 + \cdots + s_l = k$.

Thus $\lim_{N\to\infty} \#\{S \in T : [S] = \sigma\} E(\sigma)/(N^{k/2+1}\mu_N(2)^{k/2}) = L_\sigma$. In addition, note that any term with fewer than $b+1$ distinct indices $i_1, \ldots, i_k$ contributes nothing to the sum, since there are at most $O(N^b)$ of these for each equivalence class σ. Thus if we can compute how many terms there are in σ with $b+1$ distinct indices, then we will have a formula for L_σ, and therefore for the k-th moment of the expected distribution.

Proposition 4. *Every equivalence class $\sigma \in \mathcal{S}_{N,k}$ containing matrices with at least one odd entry has $L_\sigma = 0$.*

Proof. First we will construct a geometric representation of each of the terms in the sum (2).

Definition 5. An *Eulerian cycle* is a sequence of vertices in a graph that satisfies the following conditions:

(i) The length of the sequence is the number of edges in the graph.

(ii) The first vertex in the sequence is the same as the last one.

(iii) If two vertices i, j appear consecutively in the sequence then i and j are connected by an edge.

(iv) Two vertices i, j appear consecutively in the sequence (in either order) exactly n times if and only if there are exactly n edges connecting i to j.

Start with a graph with k vertices, numbered 1 through k. Connect vertices $j, j+1$ for all $j = 1, \ldots, k-1$ and connect vertices k and 1. The Eulerian cycle associated with this graph will be $1, 2, \ldots, k, 1$. We will construct a bijection between terms in the sum and labeled graphs with an associated Eulerian cycle. Fix a term B in the sum and do the following: if, in B, $i_a = i_b$ $(a < b)$, add edges between i_a and each of the neighbors of i_b, and then delete vertex i_b and all edges connected to it (thus if i_a, i_b were consecutive we would have a self-loop). Note that we started off with a connected graph with an Eulerian cycle, and that this process preserves both the connectivity and the Eulerian cycle. Call the (graph,Eulerian cycle) pair that results after no more iterations of this process can be made the *graph* of B.

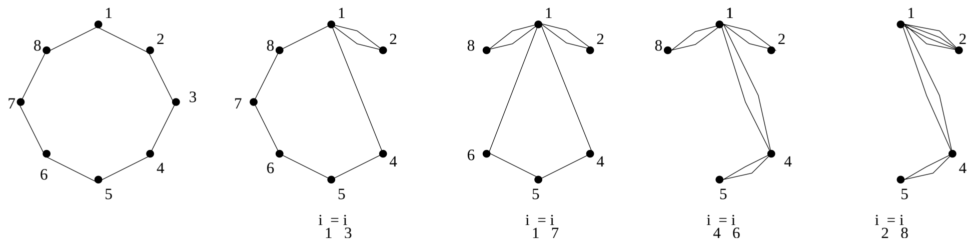

Figure 1: Example: $k = 8$, Term $= a_{12}a_{21}a_{14}a_{45}a_{54}a_{41}a_{12}a_{21}$
$i_1 = 1$ $i_2 = 2$ $i_3 = 1$ $i_4 = 4$ $i_5 = 5$ $i_6 = 4$ $i_7 = 1$ $i_8 = 2$
Beginning Cycle: 1,2,3,4,5,6,7,8,1. Final Cycle: 1,2,1,4,5,4,1,2,1.

Now consider a labeled connected graph G with k edges and an Eulerian cycle. Follow the Eulerian cycle around the graph, adding a_{ij} to the term for every edge transversed from vertex i to vertex j. This is clearly the inverse of the above transformation; thus we see that we have a bijection between terms in the sum and the pairs of (graph, Eulerian cycle).

Now consider an equivalence class $\sigma \in S_{N,k}$, which is defined by b nonzero entries $s_1, \ldots, s_b$. This means that every graph of a term in σ will have b pairs of connected vertices. In addition, a term in σ will contribute to the moment if and only if it has $b+1$ distinct indices. Thus there will be $b+1$ vertices in the graph of the term (in particular, note that there can't be any self-loops). For this to be true the graph must be a tree with some edges doubled, tripled, etc. For this tree to have an Eulerian cycle, all edges must have even multiplicity. Thus each of the s_i must have been even, and this completes the proof of proposition 4. □

Corollary 6. *All odd moments of the distribution are* 0*.*

Proof. Consider an equivalence class $\sigma \in \mathcal{S}_{N,k}$ for k odd. This will have b nonzero entries with $\sum_{i=1}^{b} s_i = k$. Thus one of the s_i must be odd, so $L_\sigma = 0$. Since σ was arbitrary, we see that the moment will be 0. □

4 The Moment Formula

Definition 7. Let V_k be the set of all $(e_1, \ldots, e_l)$ such that

(i) $\sum_{i=1}^{l} e_i = k$.

(ii) $e_1 \geq e_2 \geq \cdots \geq e_l > 0$.

Suppose $\{c_0, \ldots, c_l\}$ is a set of $l+1$ distinct colors, and define $T((e_1, \ldots, e_l))$ to be the number of colored rooted trees with $k+1$ vertices satisfying the following conditions:

(i) There are exactly e_i nodes of color c_i. The root node is the only node colored c_0.

(ii) If nodes a and b are the same color then the distance from a to the root is the same as the distance from b to the root.

(iii) If nodes a and b are colored equal colors then their parents are also colored equal colors.

Theorem 2. *The $2k$-th moment of the distribution of normalized eigenvalues is*

$$\sum_{(e_1,\ldots,e_l)\in V_k} T((e_1, \ldots, e_l)) \prod_{i=1}^{l} C_{2e_i}. \tag{4}$$

Proof. We know that any term associated with a matrix with an odd entry does not contribute to the sum. Thus we will only consider the terms associated with matrices with only even entries.

Consider any term in (3). The order coefficient will be $L_\sigma \prod_{i=1}^{b} C_{s_i}$. Thus to prove the theorem all we need to do is calculate L_σ.

Consider the graph of a term of the $2k$-th moment with b entries in the associated matrix. Consider vertices i and j that are connected by n edges. This represents that the variable a_{ij} appears n times in the corresponding sum. Now let $(b_1, \ldots, b_l)$ be such that $\sum_{i=1}^{l} b_i = 2k$, all b_i are even, and $b_1 \geq \cdots \geq b_l > 0$. Consider all graphs of terms in (2) with $l+1$ vertices and b_1 edges between two vertices, b_2 between two others, etc. We construct a bijection between these and rooted colored trees with $k+1$ vertices. Fix such a term B. Color each vertex a different color. We will construct a tree corresponding to this Eulerian cycle using the following algorithm. Mark node n, the beginning of the Eulerian cycle, as the current node. Create a root node and color it the same color as n. Make a step in the Eulerian cycle. If we have moved between nodes i and j (in either direction) an even number of times, mark the parent of the current node current. If we have moved between nodes i and j an odd number of times, add a rightmost child to the current node, color it the same color as vertex j, and mark it current. Repeat the above for each step in the Eulerian cycle.

Clearly, this is injective. Note that $b_i/2$ times we will create a node of the same color. We will end up with a tree with $k+1$ nodes, with $b_1/2$ of the colored one color, $b_2/2$ of them colored another color, etc., so condition (i) will be satisfied. Also, since the nodes of one color will all be created by going from one vertex of the same color, condition (iii) will be satisfied. Lastly, notice that because the graph of each term we are considering is a tree (with some multiple edges) each node is a well-defined distance from the root. This will be preserved by the algorithm (since we go up the tree when we decrease the distance to the root and down the tree when we increase it) so condition (ii) will be satisfied.

Now consider a tree satisfying conditions (i)-(iii). Execute the following algorithm: if we have two nodes i and j of the same color, draw edges between i and each of the neighbors of j. Then delete j and all of the edges from it. Repeat until there are no more nodes of equal colors, and then double all of the edges. This creates a graph with b_1 edges between two vertices, b_2 between two others, etc. In addition, if the original tree is transversed from the root from left to right, going along each edge twice, this will create an Eulerian cycle of the end graph. This is clearly the reverse of the original algorithm. Thus it is a bijection, and there will be $T((b_1/2, \ldots, b_l/2))$ graphs with coefficient $\prod C_{b_i}$. Letting $e_i = b_i/2$ we obtain the desired formula. □

5 Weak Convergence to the Distribution

We have shown that the moments of the distribution converge to the moments given in the moment formula. However, we have not yet shown that for any specific matrix the distribution of eigenvalues will be close to a distribution with the given moments. To have a weak convergence, we need to show the following:

Proposition 8. *The variances of the moments tends to* 0*. In other words,*

$$\lim_{N\to\infty} \left(\mathbb{E}[\mathbb{E}[x^k]_{A_N}^2] - \mathbb{E}[\mathbb{E}[x^k]_{A_N}]^2\right) = 0.$$

Proof. When we square the sum (1) we multiply terms in pairs. For any two terms this simply means that we add the exponents on the entries with the same index; thus we simply add the two associated matrices. Consider a term A in the square, the product of terms A_1 and A_2 in (1). Suppose S_i is the matrix associated to S_i (which has b_i nonzero entries), and that $S = S_1 + S_2$ has b nonzero entries $s_1, \ldots, s_b$. Clearly, the largest number of distinct indices in S will occur if and only if no entries in S_1 share an index with S_2, and vice versa; then the number of distinct indices in the term is at most $b_1+1+b_2+1 = b+2$. We can do an analysis analogous to that of section 3 to group the terms in the square of (1) by equivalence class of associated matrix, and see that the only terms that contribute to the moment are those with $b+2$ distinct indices.

When we multiply two terms of (1) we overlap the graphs of the terms. The number of indices will be the number of nodes in the graph. In addition, notice that this graph need no longer be connected; it can have two components. For an equivalence class σ of matrices with b nonzero terms in the square of (1) we need to count the (graph, Eulerian cycle) pairs that have b pairs of connected nodes and $b+2$ distinct nodes; this is only possible if we have two components to the graph. Thus we simply need to count the number of ways of picking two disjoint (graph, Eulerian cycle) pairs. But this is exactly the square of the moment formula, which are exactly the terms canceled out by $\mathbb{E}[\mathbb{E}[x^k]_{A_N}]^2$ in the desired equation. Thus we see that

$$\mathbb{E}[\mathbb{E}[x^k]_{A_N}^2] - \mathbb{E}[\mathbb{E}[x^k]_{A_N}]^2 = O\left(\frac{1}{N}\right).$$

Letting N tend to infinity we obtain the desired result. □

6 Implications of the Moment Formula: Proof of Theorem 1

Proposition 9. *The distribution of eigenvalues is symmetric.*

Proof. This is immediate from corollary 6, as a distribution is symmetric if and only if all of its odd moments are 0. □

Proposition 10. *If $C_k = O(\alpha^k)$ for some constant α, then the distribution of eigenvalues is uniquely determined by its moments.*

Proof. Note that for each tree there are at most $k!$ ways of coloring it, since if we take e_1 nodes of one color, e_2 of another, etc. and then rearrange the colors among all of the nodes in all of the possible ways, the legal colorings will be a subset of that. We know that the number of rooted trees with $k+1$ nodes is the k-th Catalan number (see [3] for more details). Since the k-th Catalan number is $\frac{1}{k+1}\binom{2k}{k}$, $T((e_1,\ldots,e_l)) < \frac{k!}{k+1}\binom{2k}{k} = \frac{(2k)!}{(k+1)!} < 2^k k^k$. In addition, the number of sorted partitions of k is smaller than the number of unsorted partitions of k, which equals $\sum_{j=1}^{k}\binom{k-1}{j-1} = 2^{k-1} < 2^k$ (since the number of ways of partitioning k into j positive partitions is $\binom{k-1}{j-1}$ for $j = 1,\ldots,k$).

Then

$$\begin{aligned} k\text{th moment} &= \sum_{(e_1,\ldots,e_l)\in V_k} T((e_1,\ldots,e_l)) \prod_{i=1}^{l} C_{2e_i} \\ &= O\left(\alpha^k \sum_{(e_1,\ldots,e_l)\in V_k} T((e_1,\ldots,e_l))\right) \\ &= O((4k\alpha)^k), \end{aligned}$$

where the last step used that $\sum 2e_i = k$.

A probability density is uniquely determined by its moments $\{\mu(k)\}$ if all the $\mu(k)$ are finite and if the power series $\sum_k \mu(k) r^k/k!$ has positive radius of convergence (see Theorem 30.1 of [1], for example). Plugging in the above estimate for the k-th moment, we see that $\sum_k \mu(k) r^k/k!$ is bounded above by

$$\sum_{k=1}^{\infty} \frac{(4k\alpha)^k r^k}{k!},$$

which has a positive radius of convergence $r < 1/(4\alpha e)$, so the distribution will be determined by its moments. ☐

Note that this agrees with Wigner's law. For Wigner's law, $p_1 = p_2 = \cdots$ (so $\mu_N(k) = \mu_M(k)$ for all M, N). For $k > 2$

$$C_k = \lim_{N\to\infty} \frac{\mu_N(k)}{N^{k/2-1}\mu_N(2)^{k/2}} = \frac{\mu_N(k)}{\mu_N(2)^{k/2}} \lim_{N\to\infty} N^{1-k/2} = 0.$$

Thus proposition 10 is applicable in the case when all of the distributions are the same and have finite moments, and the distribution of eigenvalues is determined by its moments.

Proposition 11. *If $C_{2m} = 0$ for all $m > 1$ then the distribution is a semicircle.*

Proof. If only C_2 is nonzero then all of the coefficients of $T(\cdot)$ in (4) are 0 except for the one of $T((1,1,\ldots,1))$. Clearly, $T((1,1,\cdots,1))$ is the number of rooted trees (since all of the colors are different and interchangeable). We saw that this is the k-th Catalan number; these are the moments of the distribution $\frac{1}{2\pi}\sqrt{4-x^2}$, which under renormalization becomes the semicircle (see [13]). $\square$

Proposition 12. *If $C_{2m} > 0$ for some $m > 1$ then the distribution of eigenvalues has unbounded support.*

Proof. Suppose that $C_{2m} > 0$ for some $m > 1$. We find a lower bound for the moments of the distribution, and show that they grow faster than an exponential. Since a bounded distribution implies at most exponential moment growth, this will show that the support of the distribution is unbounded.

We will find a lower bound for all moments of the form $k = 2m\ell$, $\ell \in \mathbb{N}$. Since all of the terms in (4) are nonnegative, we will only look at one partition, $(m, m, \ldots, m)$. In addition, we will only look at one tree out of all of the trees: the one where all nodes are the direct children of the root. This tree can be colored with the requested colors in

$$\frac{1}{\ell!}\binom{m\ell}{m}\binom{m\ell-m}{m}\cdots\binom{m}{m}$$

ways, since we need to choose which nodes are the same color, but it does not matter which color is which. Thus the $2m\ell$-th moment is larger than

$$\frac{(m\ell)!}{(m!)^\ell \ell!}C_{2m}^\ell > \left(\frac{C_{2m}}{m!}\right)^\ell (m\ell-1)!!,$$

which grows faster than exponential. Thus the moments grow faster than exponential, which implies that the distribution has unbounded support.

$\square$

7 Two Applications

7.1 The Truncated Distribution $\frac{C}{1+|x|^m}$

We consider a distribution with infinite higher moments by truncating it at increasingly large bounds. Consider a probability distribution of the form

$$\frac{A}{1+|x|^m} \qquad m > 1$$

(note that $m = 2$, $a = \frac{1}{\pi}$ is the Cauchy distribution). For an increasing function $f : \mathbb{N} \to \mathbb{R}_+$ we define a distribution p_N by

$$p_N(x) = \begin{cases} 0 & \text{if } |x| > f(N) \\ \frac{A_N}{1+|x|^m} & \text{otherwise,} \end{cases}$$

where $A_N = \left(\int_{-f(N)}^{f(N)} \frac{dx}{1+|x|^m} \right)$. The moments of this distribution are

$$\mathbb{E}[x^n] = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \Theta(1) & \text{if } n \text{ even, } n < m-1 \\ \Theta(\log f(N)) & \text{if } n \text{ even, } n = m-1 \\ \Theta(f(N)^{n+1-m}) & \text{if } n \text{ even, } n > m-1 \end{cases} \tag{5}$$

where a function f is $\Theta(g)$ if $f = O(g)$ and $g = O(f)$. We have three cases.

If $m > 3$ then the second moment of the distribution is

$$\int_{-f(N)}^{f(N)} \frac{x^2}{1+|x|^m}\, dx < \int_{-\infty}^{\infty} \frac{x^2}{1+|x|^m}\, dx \ll \int_{-\infty}^{\infty} \frac{1}{1+|x|^{1+\epsilon}}\, dx,$$

which is bounded. Thus for $C_{2k} > 0$ we would need $\mu_N(2k) = \Theta(N^{k-1})$, so

$$f(N)^{2k+1-m} \asymp N^{k-1} \Longrightarrow f(N) \asymp N^{(k-1)/(2k+1-m)}.$$

However, since $\frac{x-1}{2x+1-m}$ is a decreasing function with lower bound $1/2$ and $f(N)$ is independent of k, we see that $f(N) = O(N^{1/2+\epsilon})$ for all $\epsilon > 0$ for C_{2k} to exist for all k, and then $C_{2k} = 0$.

Now suppose $m = 3$. From (5) we see that the second moment of the distribution grows as $\Theta(\log f(N))$ and the higher moments grow as $\Theta(f(N)^{n-2})$. Thus in order for all C_{2k} to be finite we need

$$\frac{\mu_N(k)}{N^{k/2-1}\mu_N(2)^{k/2}} = \frac{f(N)^{2k-2}}{N^{k-1}\log^k f(N)} = O(1),$$

so

$$f(N) = O(N^{1/2} \log^{k/(2k-2)} f(N))$$

for all k. However, since $k/(2k-2)$ is a decreasing function of k and $f(N)$ is independent of k we see that this implies

$$f(N) = O(N^{1/2} \log^{1/2+\epsilon} f(N))$$

for all $\epsilon > 0$. Since $f(N) = o(N^{1/2} \log^{k/(2k-2)} f(N))$ for all k, all C_{2k} with $k > 1$ are 0. Thus the distribution will be a semicircle (see [6] for a computation of the moments of the semicircle).

Now suppose $1 < m < 3$. Then all even moments are $\Theta(f(N)^{n+1-m})$. For C_{2k} to be finite we need

$$\frac{f(N)^{2k+1-m}}{N^{k-1} f(N)^{k(3-m)}} = O(1)$$

which means that $f(N) = O(N^{1/(m-1)})$. Note that if $f(N) = o(N^{1/(m-1)})$ then for all $k > 1$, $C_{2k} = 0$ (so the distribution of moments will tend to a semicircle), while if $f(N) = \Theta(N^{1/(m-1)})$, C_{2k} will be positive. In addition, from a simple integration we see that $C_{2k} = 1/(2k+1-m)$. Thus the formula for the moments when $f = \Theta(N^{1/(m-1)})$ will be

$$\sum_{(e_1,\ldots,e_l)\in V_k} T((e_1,\ldots,e_l)) \prod_{i=1}^{l} (2e_i + 1 - m)^{-1}.$$

In this case we see from propositions 10 and 12 that this distributions will be uniquely determined by its moments and will have unbounded support.

7.2 Approximately k_N-regular Graphs

We would like to be able to apply the theory that we developed to combinatorial constructions, such as large regular graphs. We know that the eigenvalue distributions of adjascency matrices of such graphs follow McKay's law (see [4]); it would be interesting to obtain McKay's law applying the methods used above. However, we cannot do this because we are choosing matrices via probability distributions of their entries, not via other matrix distributions. In addition, combinatorial objects such as undirected graphs do not have negative entries, so we could not assign a mean-0 probability to the entries of such a matrix.

In order to approximate the behavior of the ensemble of adjascency matrices of k-regular graphs, we take a probability distribution that is expected to produce an $N \times N$ matrix with k_N nonzero entries in each row and column. Each of these entries can be ± 1 with equal probability, so that we have a mean-0 distribution. To do this, we consider the following probability distribution:

$$\Pr(a_{ij} = 1) = \Pr(a_{ij} = -1) = \frac{k_N}{2(N-1)}.$$

if $i \neq j$. If $i = j$, $a_{ij} = 0$. (From the earlier calculations of moments it is clear that in this case the formulas for the moments will still be valid.) Then we know that

$$\mathbb{E}[x^\ell] = \begin{cases} \frac{k_N}{N-1} & \text{if } \ell \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$C_\ell = \lim_{N\to\infty} \frac{k_N/(N-1)}{N^{\ell/2-1}(k_N/(N-1))^{\ell/2}} = \lim_{N\to\infty} k_N^{1-\ell/2}.$$

Thus if $k_N \to \infty$ as $N \to \infty$, $C_\ell = 0$ for $\ell > 2$, so we have a semicircle distribution. Otherwise, if $\lim_{N\to\infty} k_N = k$, the support of the distribution will be unbounded, so it will not be the semicircle. The odd moments of the distribution will be 0, and the even moments will be given by

$$\mathbb{E}[\mathbb{E}[x^{2\ell}]] = \sum_{(e_1,\ldots,e_m)\in V_\ell} T((e_1,\ldots,e_m))k^{m-\ell}.$$

8 Acknowledgements

I would like to thank the American Institute of Mathematics for hosting the REU during which this work was done, and Steven J. Miller for suggesting the problem, discussing it with me, and editing multiple copies of this paper. I also owe large parts of this work to Ilya Zakharevich for his suggestions about generalizations and Gregory Price for giving me the idea of looking at graphs. Special thanks go to Mark Adler who read this and told me to figure out exactly what it was that I proved.

References

[1] P. Billingsley, *Probability and Measure*, 2nd ed., (John Wiley & Sons, 1986).

[2] E. Bogomolny, O. Bohigas, M.P. Pato. "Distribution of eigenvalues of certain matrix ensembles." Phys. Rev. **55**, 6707-6718. 1996.

[3] J. Borwein and D. Bailey. *Mathematics by Experiment: Plausible Reasoning in the 21st Century*. Natick, MA: A. K. Peters, 2003.

[4] B. McKay. "The expected eigenvalue distribution of a large regular graph." J. Lin. Alg. Appl. **40**, 203-216, (1981).

[5] M. L. Mehta, *Random Matrices*, 2nd ed., (Academic Press, 1991).

[6] S. J. Miller and R. Takloo-Bighash. *An Invitation to Modern Number Theory* (Princeton University Press, to appear).

[7] L. A. Pastur. "On the Spectrum of Random Matrices." *Theoret. and Math. Phys.*,**10**, 67-74, (1972).

[8] A. Soshnikov. "Universailty at the edge of the spectrum in Wigner random matrices." *Commun. Math. Phys.*, **207**, 697-733, (1999).

[9] A. Soshnikov. "Poisson Statistics for the Largest Eigenvalues of Wigner Random Matrices with Heavy Tails." arXiv:math.PR/0405090.

[10] A. Soshnikov, Y. Fyodorov. "On the Largest Singular Values of Random Matrices with Independent Cauchy Entries." *Elec. Commun. Probab.*, **9**, 82-91, (2004).

[11] C.A. Tracy, H. Widom. "Level-spacing distributions and the Airy kernel." *Commun. Math. Phys.*, **159**, 151-174, (1994).

[12] C.A. Tracy, H. Widom. "On orthogonal and symplectic random matrix ensembles." *Commun. Math. Phys.*, **177**, 724-754, (1996).

[13] E. Wigner. "Characteristic Vectors of Bordered Matrices with Infinite Dimensions." Ann. of Math., **62**, 548-564.

[14] E. Wigner. "On the Distribution of the Roots of Certain Symmetric Matrices." Ann. of Math., **67**, 325-327. 1957.